Annals of Optimization with Applications

www.anowa.reapress.com

Ann. Optim. Appl. Vol. 2, No. 1 (2025) 57–70.

Paper Type: Original Article

Ranking Efficient DMUs in DEA Based on System-Wide Performance: Directional Distance Function Approach

Israfil Roshdi[1,*], Mahmoud Mehdiloozad[1]

[1] Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran; i.roshdi@gmail.com; m.mehdiloozad@gmail.com.

Citation:

Received: 17April 2024 *Revised: 21 July 2024* *Accepted: 28 September 2024*	Roshdi, I., & Mehdiloozad, M. (2025). Ranking efficient DMUs in DEA based on system-wide performance: Directional distance function approach. *Annals of optimization with applications*, 2(1), 57-70.

Abstract

In Data Envelopment Analysis (DEA), efficiency score can be a criterion for ranking inefficient Decision Making Units (DMUs). However, this criterion cannot rank efficient DMUs. This research proposes a methodology for ranking extreme and nonextreme efficient DMUs. The proposed method, based upon the influence of the individual DMUs' performances on the system-wide performance, has many attractive properties and successfully overcomes some difficulties in ranking methods. First, a so-called Directional Slack-Based Measure (DSBM) is presented to measure system-wide performance using the directional distance function. Then, by employing this measure, a so-called system-wide DSBM is developed that is a generalization of the approach presented by Cooper et al. [22]. The main idea behind our proposed method is that the more the omission DMU_a has influence (increase) on the system-wide performance, the better DMU_a performance. Two illustrative examples compare the proposed method with other ranking methods.

Keywords: Data envelopment analysis, Efficiency, Ranking, Directional distance function, Directional slack-based measure, System-wide directional slack-based measure.

1 | Introduction

In evaluating Decision Making Units (DMUs) using the Data Envelopment Analysis (DEA) technique, initially proposed by Charnes et al. [1], DMUs are classified successfully into efficient DMUs and inefficient DMUs. Efficient DMUs are identified by an efficiency score equal to 1, and inefficient DMUs have efficiency scores less than one. Although efficiency score can be a criterion for ranking inefficient DMUs, this criterion

Corresponding Author: i.roshdi@gmail.com

cannot rank efficient DMUs. In such cases, it is very important to have a tool to discriminate between the efficient DMUs and rank them. That is why many types of research for ranking the efficient units were published in DEA literature, and many models were formulated. Sexton et al. [2] first proposed the cross-efficiency method to rank efficient DMUs in this area. After that, a great variety of ranking methods for more details [3], [4].

With different properties have been proposed, e.g., super efficiency technique, Benchmark method, ranking by l_1-norm, tchebycheff norm and l_2-norm, ranking by common set of weights, monte carlo method, ranking by using TOPSIS method, and so on, to rank DMUs [5–15]. However, each has advantages and disadvantages that can be used in a specific area according to these features. Hence, none could completely solve the ranking problem and be selected as the best methodology.

This paper presents a new ranking method based on system-wide efficiency that differs utterly from existing ranking methods. In our framework, firstly, using the directional distance function, a recent concept due to Luenberger [16], [17] and Chambers et al. [18], [19], we will extend a so-called Directional Slack-Based Measure (DSBM) of efficiency where the Enhanced Russell Measure (ERM) and Slack-Based Measure (SBM) are special cases of it [20], [21].

Secondly, using this measure, we will develop a so-called System-wide Directional Slack-Based Measure (SDSBM) that is a generalization of the approach presented by Cooper et al. [22]. Finally, our work leads to a ranking index based on the influence of the individual DMUs' performances on the system-wide performance.

Therefore, in contrast to the traditional ranking models that compare DMUs with the best performances (Pessimistic) or the worst performances (Optimistic), our method implicitly compares DMUs with both the best and worst performances. Furthermore, our proposed method has many desirable properties and advantages; some problematic areas in ranking efficient DMUs, e.g., infeasibility and instability in the super-efficiency technique, do not occur.

The remainder of this paper unfolds as follows. In the next section, we introduce the DSBM of efficiency and provide a detailed discussion about the properties and features of this measure. Further, employing it, we generalize the approach Cooper et al. [23] presented and develop a DSBM of system-wide efficiency.

In Section 3, we present our ranking approach and talk about its methodology and properties. In Section 4, by introducing two illustrative examples, the proposed method is compared with other ranking methods. Finally, concluding remarks and the directions for future research are collected in the last section.

2 | Aggregation in Data Envelopment Analysis with Directional Slack-Based Measure of Efficiency

This section presents a new DSBM of efficiency and then discusses its properties. Next, we review concepts and the model for system-wide efficiency measurement in DEA, proposed by Cooper et al. [23]. Finally, we employ the DSBM to develop a new SDSBM.

2.1 | Directional Slack-Based Measure of Efficiency

Throughout this paper, we deal with n DMUs with m inputs $(i=1,...,m)$ and s outputs $(r=1,...,s)$. The input and output vectors of DMU_j $(j=1,...,n)$, are

$$x_j = (x_{1j},...,x_{mj})^T,\ y_j = (y_{1j},...,y_{sj})^T,$$

and where $x_j \geq 0$, $x_j \neq 0$, $y_j \geq 0$ and $y_j \neq 0$. The Production Possibility Set (PPS), T, is the set of all feasible input and output vectors, and it is defined as follows:

$$T = \{(x, y): \text{ x can produce y}\}. \tag{1}$$

Charnes et al. [1] have deduced the following unique PPS, constructed from a set of n observed DMUs, considering the inclusion of observations, convexity, ray unboundedness, and free disposability of inputs and outputs postulates. This set is denoted by T_c the prevalence of the production technology's constant returns to scale assumption.

$$T_c = \left\{(x, y) \middle| x \geq \sum_{j=1}^{n} \lambda_j x_j \,,\; y \leq \sum_{j=1}^{n} \lambda_j y_j \,,\; \lambda_j \geq 0,\; j = 1, 2, ..., n\right\}. \tag{2}$$

For our purpose, we specifically use the directional distance function concept. The directional distance function, recently introduced by Chambers et al. [18], [19] is a version of Luenberger's shortage function, which generalizes the traditional Shephard distance function and plays a significant role in production theory and is well-suited to the task of providing a measure of technical efficiency in the whole input-output space [16], [17], [24].

Considering the structure of T_c, if the direction vector $g = (g^-, g^+)$ has been selected such that

$$\underset{i}{Max}\left\{\frac{x_{ij}}{g_i^-}\right\} \leq 1,\; j = 1, ..., n, \tag{3}$$

E.g., one can apply the following direction vectors:

$$g_i^- = x_{io},\; g_r^+ = y_{ro},\; \text{for all } i, r, \tag{4}$$

$$g_i^- = \overline{x}_i = \underset{j}{Max}\{x_{ij}\},\; g_r^+ = \overline{y}_r = \underset{j}{Max}\{y_{rj}\},\; \text{for all } i, r, \tag{5}$$

then the DEA formulation of DSBM relative to *Eq. (2)* is as follows:

$$\begin{aligned} e_o = Min \quad & E_o = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \beta_i^-}{1 + \frac{1}{s}\sum_{r=1}^{s} \beta_r^+}, \\ s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io} - \beta_i^- g_i^-, \quad i = 1, ..., m, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro} + \beta_r^+ g_r^+, \quad r = 1, ..., s, \\ & \beta_i^- \geq 0,\; i = 1, ..., m, \\ & \beta_r^+ \geq 0,\; r = 1, ..., s, \\ & \lambda_j \geq 0, \quad j = 1, ..., n. \end{aligned} \tag{6}$$

The vector $g = (g^-, g^+)$ represents the pre-assigned direction vector along which DMU_o, if it is an interior point, it is projected onto the efficient frontier of the PPS. Here β_i^- β_r^+, it means the rates of contraction and expansion in the ith input and rth output DMU_o it has projected onto the efficient frontier of T_c the direction g. Furthermore, the objective *Eq. (6)* jointly maximizes the values of β_i^- $(i = 1, ..., m)$, β_r^+ $(r = 1, ..., s)$ and. Since

$$\left(\lambda_j = 0, \text{ for all j, } j \neq o,\; \beta_i^- = \beta_r^+ = 0, \text{ for all i, r}\right).$$

It is a feasible solution of this model, always $e_o \leq 1$. In addition, since

$$x_{io}-\beta_i^- g_i^- \geq 0, \ (i=1,\ldots,m), \ \beta_i^- \leq \frac{x_{io}}{g_i^-}, \ (i=1,\ldots,m).$$

Therefore, according to *Model (3)*, $0 \leq e_o \leq 1$ e_o it can be interpreted as an efficiency measure. The optimal value of *Eq. (4)* e_o is the efficiency score of DMU_o, and based on it, we determine a DMU as being DSBM-efficient as follows:

Definition 1. DMU_o is said to be DSBM-efficient if and only if $e_o = 1$.

This condition is equivalent to $\beta_i^{-*} = \beta_r^{+*} = 0$, for all i, r, in each optimal Solution of *Eq. (6)*, i.e., there is no input inefficiency (Waste) and no output inefficiency (Shortfall) in all inputs and outputs in any optimal solution.

Remark 1. The ERM and SBM models are special cases of *Model (6)*. After assigning the direction *Vector (7)*, if we put $\theta_i^- = 1-\beta_i^-$, $\varphi_r^+ = 1+\beta_r^+$, for all i, r, ERM is derived, and if we put $s_i^- = \beta_i^- x_{io}$, $s_r^+ = \beta_r^+ y_{ro}$, for all i, r, SBM is derived.

Theorem 1. DMU_o is DSBM-efficient (By assigning any positive direction vector) if and only if it is ERM-efficient.

Proof: This is obvious. By allocating a suitable direction vector, DSBM will have many attractive properties that we outline as follows:

Computational aspect

This model is a fractional programming problem. However, it can be transformed into a linear program utilizing the Charnes–Cooper transformation, similar to the ERM model [25]. Moreover, due to their equivalency, we can generate an optimal solution for the corresponding DSBM model from an optimal solution of the LP form.

Completeness

This measure is 'complete' in that it is non-oriented, contrasting with oriented measures. It considers all inefficiencies associated with non-zero slacks that the model may identify. Therefore, further discrimination is obtained because of a non-radial movement toward the efficient frontier.

Unit invariance

This model will be unit invariant by selecting a direction vector such that the i[th] component g_i^- $(i=1,\ldots,m)$ and i[th] component g_r^+ $(r=1,\ldots,s)$ have the same units of measurement as the i[th] input and r[th] output. The same values e_o will be obtained with any unit of measure employed for inputs and outputs. For instance, the direction *Vectors (5)* and *(6)* satisfy this condition.

Monotonicity

The measure is strictly monotone, decreasing in each β_i^- β_r^+.

Decision maker's preferences incorporation

In some practical cases, if the Decision Maker (DM) does not prefer the efficient units equally, it is necessary to consider the DM's judgments or a priori knowledge. A remarkable property of *Model (6)* is that by choosing a suitable direction vector, g, the DM's preference information can be explicitly considered account. In this way, unrealistic weighting schemes that might be used by the DMUs are eliminated, and, therefore, the target obtained will be more meaningful than the usual target obtained for conventional DEA models.

We can flexibly modify vectors according to the preference orders of inputs/outputs given by DM g. Indeed, the values of the modified direction vector g''s components describe the relative importance of inputs/outputs given by DM. Let the non-zero weights w_i $(i=1,...,m)$ v_r $(r=1,...,s)$ be associated with the priorities DM gives to the inputs and outputs, respectively, such that the larger the w_i (v_r), the more important the i[th] input (r[th] output) is. After incorporating these weights in *Eq. (6)*, the coefficient of variables β_i^- β_r^+ and the objective function will be w_i, v_r and, respectively. Therefore, the components of the modified direction vector g' should be

$$g_i^{-\prime}=\xi_i g_i^-,\ g_r^{+\prime}=\psi_r g_r^+,$$

where

$$\xi_i=\frac{1}{w_i},\ \psi_r=\frac{1}{v_r}.$$

This shows that if an input (output) is of greater importance, it should be attached to a larger weight or, equivalently, a small direction component. By considering *Eq. (3)*, we must have $\xi_i \geq 1$ $i=1,...,m$ equivalently $w_i \leq 1$, $r=1,...,s$ [1] We will elaborate on this in Section 7.

Flexibility in computer programming

Another advantage of this model is that we only need to change the direction vector's inputs in this program by writing its computer code to achieve new efficiency scores concerning the new direction. Furthermore, running this program for some directions can calculate the average obtained scores as a final score for a given DMU.

Extension to hybrid models

One can extend the proposed model to its hybrid form [22].

Extension to models with arbitrary returns to scale

By adding the constraint

$$L \leq \sum_{\substack{j=1 \\ j\neq o}}^{n}\lambda_j \leq U,\ (0\leq L\leq 1,\ 1\leq U\leq \infty).$$

To the constraints of this model, one can easily exert the relaxed convexity condition for different types of returns to scale to them. For instance, if $L=U=1$ the corresponding PPS is satisfied in variable returns to scale assumption, in the following subsection, we will employ *Model (6)* to provide a directional distance-based model for measuring system-wide efficiency.

2.2 | A Generalized System-Wide Performance Measure in Data Envelopment Analysis

First, we consider the collection of n observed DMUs as a system, (x_T, y_T) where

$$x_{iT}=\sum_{j=1}^{n}x_{ij},\ i=1,...,m,$$

and

[1] If the given weights do not satisfy these conditions, their normalized (dividing by $\text{Max}\{w_i : i=1,...,m\}$) form will fulfill them.

$$y_{rT} = \sum_{j=1}^{n} y_{rj},\ r = 1,...,s.$$

Th x_{iT} y_{rT} ey are the sums of the individual DMUs inputs and outputs, respectively. Cooper et al. [22] proposed the following DEA model to measure the system-wide performance:

$$\begin{aligned} \rho_T = \text{Min} \quad & \frac{\frac{1}{m}\sum_{i=1}^{m}\delta_{iT}}{\frac{1}{s}\sum_{r=1}^{s}\eta_{rT}}, \\ \text{s.t.} \quad & \sum_{j=1}^{n}\lambda_j x_{ij} + \lambda_T x_{iT} \le \delta_{iT} x_{iT}, \qquad i = 1,...,m, \\ & \sum_{j=1}^{n}\lambda_j y_{rj} + \lambda_T y_{rT} \ge \eta_{rT} y_{rT}, \qquad r = 1,...,s, \\ & 0 \le \delta_{iT} \le 1,\ i = 1,...,m, \\ & \eta_{rT} \ge 1, \qquad r = 1,...,s, \\ & \lambda_T \ge 0,\ \lambda_j \ge 0,\ \ j = 1,2,...,n. \end{aligned} \tag{7}$$

They proved that λ_T its coefficients can be deleted in *Eq. (7)*.

Similar to *Model (6)*, if the direction vector $g = (g^-, g^+)$ has been selected such that

$$\underset{i}{\text{Max}}\left\{\frac{x_{iT}}{g_i^-}\right\} \le 1. \tag{8}$$

Then, the mathematical programming of the SDSBM is as follows:

$$\begin{aligned} e_T = \text{Min} \quad & E_T = \frac{1 - \frac{1}{m}\sum_{i=1}^{m}\beta_i^-}{1 + \frac{1}{s}\sum_{r=1}^{s}\beta_r^+}, \\ \text{s.t.} \quad & \sum_{j=1}^{n}\mu_j x_{ij} + \mu_T x_{iT} \le x_{iT} - \beta_i^- g_i^-, \qquad i = 1,...,m, \\ & \sum_{j=1}^{n}\mu_j y_{rj} + \mu_T y_{rT} \ge y_{rT} + \beta_r^+ g_i^-, \qquad r = 1,...,s, \\ & \beta_i^- \ge 0,\ i = 1,...,m, \\ & \beta_r^+ \ge 0,\ r = 1,...,s, \\ & \mu_T \ge 0,\ \mu_j \ge 0,\ \ j = 1,2,...,n. \end{aligned} \tag{9}$$

Through *Eq. (8)*, we observe that $0 \le e_T \le 1$ it e_T can be interpreted as a system-wide efficiency measure. For example, one can apply the following direction vectors:

$$g_i^- = x_{iT},\ g_r^+ = y_{rT},\ ,\ \text{for all } i, r, \tag{10}$$

$$g_i^- = n\overline{x}_i = n \times \underset{j}{\text{Max}}\{x_{ij}\},\ g_r^+ = n\overline{y}_r = n \times \underset{j}{\text{Max}}\{y_{rj}\},\ \text{for all } i, r. \tag{11}$$

If we assign the direction *Eq. (10)*, we will have the *Model (7)*. SDSBM inherently has the properties of the *Model (6)*. Another desirable property is that we can use the same direction vector to self-evaluate each system and compare their performances.

The above model is always feasible, and the optimal objective of *Eq. (9)* e_T is the efficiency score for the under-evaluation system (x_T, y_T), and the 'efficiency' of the system is introduced as follows.

Definition 2. The system (x_T, y_T) is said to be efficient if and only if $e_T = 1$. Similar to *Theorem 1* and *2* in Cooper et al. [22], we have the following theorems:

Theorem 2 (Alternate optima theorem). The system is efficient if and only if the following solutions constitute alternate optima: 1) $\mu_T^* = 1$ in *Eq. (9)* with all $\mu_j^* = 0$ and 2) $\mu_j^* = 0$ $(j = 1,\ldots,n)$, and $\mu_T^* = 0$.

Theorem 3. If the system is inefficient, then $\mu_T^* = 0$ in all optimum solutions.

Considering the above theorems, either system is efficient or inefficient; *Model (9)* always has an optimal solution $\lambda_T^* = 0$. Therefore, we can eliminate λ_T its coefficients in *Eq. (9)* and, therefore, use the following equivalent form to measure the system-wide efficiency:

$$
\begin{aligned}
& e_T = \text{Min} \quad \rho = \frac{1 - \frac{1}{m}\sum_{i=1}^{m}\tau_i^-}{1 + \frac{1}{s}\sum_{r=1}^{s}\tau_r^+}, \\
& \text{s.t.} \quad \sum_{j=1}^{n}\mu_j x_{ij} \le x_{iT} - \tau_i^- g_i^-, \qquad i = 1,\ldots,m, \\
& \qquad \sum_{j=1}^{n}\mu_j y_{rj} \ge y_{rT} + \tau_r^+ g_i^-, \qquad r = 1,\ldots,s, \\
& \qquad \tau_i^- \ge 0,\ i = 1,\ldots,m, \\
& \qquad \tau_r^+ \ge 0,\ r = 1,\ldots,s, \\
& \qquad \tau_j \ge 0,\ j = 1,2,\ldots,n.
\end{aligned} \tag{12}
$$

Regarding the relation between e_T and ρ_T, it can be easily proved that $e_T = 0$ if and only if $\rho_T = 0$.

3 | Our Proposed Ranking Method

In this section, based on the influences of the individual DMU's performances on the total system-wide performance, we propose a method for ranking all efficient DMUs. The main idea behind our proposed ranking method is that to rank a given efficient DMU, this unit is excluded from the system, and a new system is made. Then, the performance of the new system is measured.

As we will demonstrate, this exclusion results in the improvement of system efficiency. This is done for all efficient DMUs. Then, all efficient DMUs are ranked based on the corresponding new system efficiencies. Our ranking criterion is: The more omission of a unit improves the system-wide performance, the better the corresponding unit performs.

3.1 | Our Proposed Ranking Method

Because the presence of an efficient DMU_a causes the inefficiency of inefficient DMUs, omitting this unit may convert some of the inefficient DMUs into efficient DMUs or reduce the inefficiency of the inefficient DMUs (see *Fig. 1*). Finally, it increases the efficiency of the new system. Consequently, the more the system efficiency increases, the better DMU_a it performs.

For ranking an efficient unit, DMU_a it is first excluded from the set of all observed DMUs, and all remaining DMUs are considered a new system (see *Fig. 1*).

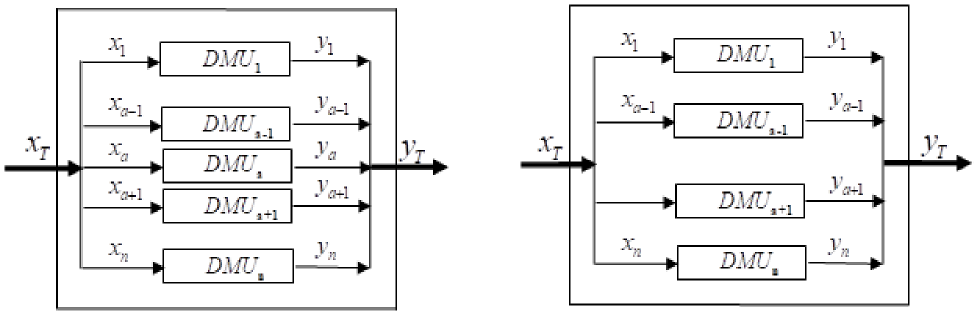

Fig. 1. The original system (Before excluding DMU_a) and the new system (After excluding DMU_o).

Secondly, to perform our approach, we evaluate the new system, $\left(x_{T^a}, y_{T^a}\right)$ with input values $x_{iT^a} = \sum_{\substack{j=1 \\ j \neq a}}^{m} x_{ij}$ $i = 1,...,m$ and output values $y_{rT^a} = \sum_{\substack{j=1 \\ j \neq a}}^{s} y_{rj},\ r = 1,...,s$, by *Model (13).*

$$
\begin{aligned}
e_{T^a} = \text{Min} \quad & E_{T^a} = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \tau_i^-}{1 + \frac{1}{s}\sum_{r=1}^{s} \tau_r^+}, \\
\text{s.t.} \quad & \sum_{\substack{j=1 \\ j \neq a}}^{n} \mu_j x_{ij} \le x_{iT^a} - \tau_i^- g_i^-, \qquad i = 1,...,m, \\
& \sum_{\substack{j=1 \\ j \neq a}}^{n} \mu_j y_{rj} \ge y_{rT^a} + \tau_r^+ g_i^-, \qquad r = 1,...,s, \\
& \tau_i^- \ge 0,\ i = 1,...,m, \\
& \tau_r^+ \ge 0,\ r = 1,...,s, \\
& \tau_j \ge 0,\ \ j = 1,2,...,n,\ j \neq a.
\end{aligned} \tag{13}
$$

It should be noted that the direction vectors assigned to *Models (12)* and *(13)* in evaluating the original and the new systems are the same. Therefore, both systems are compared using the same criteria, and the obtained targets are more meaningful.

However, if we use *Model (7)* to evaluate these systems, the original and new systems will be assessed in two different directions (x_T, y_T), $\left(x_{T^a}, y_{T^a}\right)$, which is unreasonable.

Therefore, an advantage of SDSBM over *Eq. (7)* is that several systems can be compared concerning the same criterion. Regarding the relationship between e_T and e_{T^a}, we have the following theorem:

Theorem 4. The optimal objective value of *Model (13)* in evaluating the new system e_{T^a} is not less than that of the original system, e_T $e_T \le e_{T^a}$ i.e.

Proof: Suppose that we have evaluated the new system $\left(x_{T^a}, y_{T^a}\right)$ by *Model (13)*, and we have the following optimal solution:

$$\sum_{\substack{j=1\\j\neq a}}^{n}\mu_j^*x_{ij}=x_{iT^a}-\tau_i^{-*}g_i^-, \quad i=1,\dots,m,$$
$$\sum_{\substack{j=1\\j\neq a}}^{n}\mu_j^*y_{rj}=y_{rT^a}+\tau_r^{+*}g_r^+, \quad r=1,\dots,s. \qquad \textbf{(14)}$$

Moreover, we evaluated the efficient unit $DMU_a=(x_a,y_a)$ using the *Model (6)*. Then, there exists some optimal solution of this model as follows:

$$\sum_{j=1}^{n}\lambda_j^*x_{ij}=x_{ia}, \quad i=1,\dots,m,$$
$$\sum_{j=1}^{n}\lambda_j^*y_{rj}=y_{ro}, \quad r=1,\dots,s, \qquad \textbf{(15)}$$

where

$$\lambda_a^*=1,\ \lambda_j^*=0,\ j=1,\dots,n,\ j\neq a. \qquad \textbf{(16)}$$

From *Eqs. (14)* and *(15)*, we have

$$\sum_{\substack{j=1\\j\neq a}}^{n}\mu_j^*x_{ij}+x_{ia}=x_{iT}-\tau_i^{-*}g_i^-, \quad i=1,\dots,m,$$
$$\sum_{\substack{j=1\\j\neq a}}^{n}\mu_j^*y_{rj}+y_{ra}=y_{rT}+\tau_r^{+*}g_r^+, \quad r=1,\dots,s. \qquad \textbf{(16)}$$

So,

$$\left(\mu_a'=1,\ \mu_j'=\mu_j^*,\ \text{for all j, j}\neq a,\ \tau_i^{-\prime}=\tau_i^{-*},\ \tau_r^{+\prime}=\tau_r^{+*},\ \text{for all i, r}\right).$$

Model (12) is a feasible solution for evaluating the original system. Therefore, the result is obtained. As anticipated, the above proof shows that excluding an efficient DMU will not worsen the new system's performance.

To illustrate our proposed method, consider the PPS in *Fig. 2* part (a), which consists of five DMUs: A, B, C, D, E, and F. Units A, B, and C are ERM-efficient, and units D, E, and F are ERM-inefficient.

Only unit D becomes efficient after excluding unit A, as shown in part (b). Units D and E become efficient after excluding unit B, as shown in part (c). However, by removing unit C, as shown in part (d), none of the inefficient DMUs will become efficient.

Therefore, among the efficient DMUs (A, B, and C), excluding units B and C from the observed DMUs causes the most significant and minor increase in the system-wide efficiency, respectively. Thus, units B and C are the best and worst performers among all efficient DMUs. In other words, the efficient units B and C have more and less influence on the system performance.

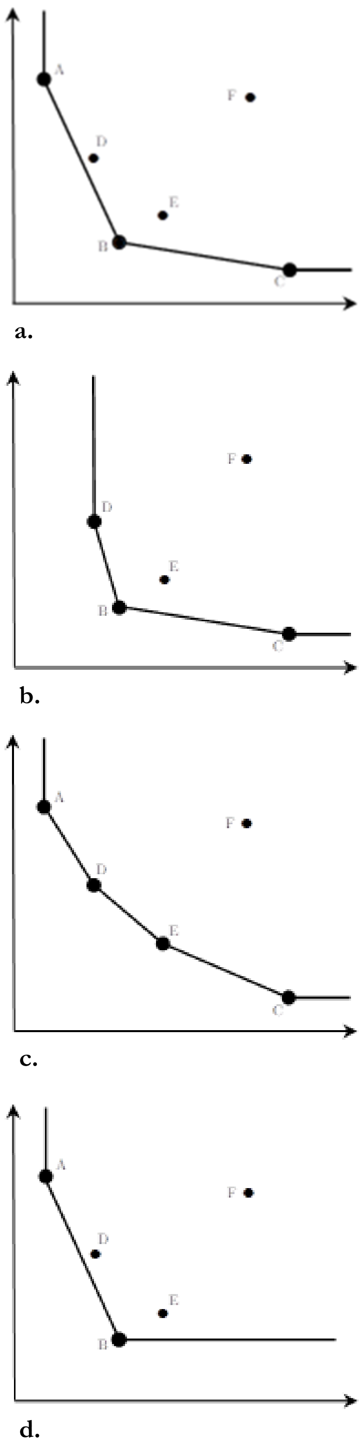

Fig. 2. Excluding units and Farrell frontiers; a. The Farrell frontier made up of all DMUs, b. The Farrell frontier after excluding unit A, c. The Farrell frontier after excluding unit B, and d. The Farrell frontier after excluding unit C.

3.2 | Theoretical and Computational Advantages of the Method

Our proposed approach is entirely different in methodology compared to all existing ranking methods. Moreover, it has several desirable properties and advantages and, in some cases, is superior to other methods, e.g., super efficiency technique. In the following, we summarize these properties:

Always feasibility

An advantage of our approach is that it guarantees the generation of feasible solutions for all efficient DMUs without any primary assumptions on data, e.g., positivity of data. Infeasibility is a common problem in almost all super-efficiency models proposed so far.

Ranking all extreme and nonextreme efficient DMUs

In contrast with some existing methods, e.g., the super efficiency technique that only ranks the extreme DMUs, our method can rank all efficient (either extreme or nonextreme) DMUs.

Incorporation of DM's preferences information into ranking

A common weakness of conventional ranking methods is that their evaluations are "value free" in the sense that they do not include DM's preferences. Now, the question arises as to whether this operation has any efficacy in ranking orders. In addition, provided the answer is positive, how can we perform it? In example *Model (2)*, we will show that the answer is positive, and using the methodology described in the property *Model (6)* of the *Model (8)*, we can do it.

Completeness

Model (9) used in our approach is also complete due to the completeness of *Model (6)*.

4 | Illustrative Examples

This section presents two numerical examples to illustrate the proposed method. By first example, we generally compare our proposed method with Cross-Efficiency-Aggressive (CEA), Cross-Efficiency-Benevolent (CEB), Benchmarking, and super-efficiency (AP model) ranking methods [5]. Using the second example, two aims are pursued:

I. Investigating the influence of taking into account DM's preferences information in ranking

II. Elaborating the strength of our method in ranking nonextreme efficient DMUs.

Example 1. The data listed in *Table 1* are extracted from Adler et al. [3]. In evaluating efficient DMUs, we apply different directions *Eqs. (10)* and *(11)*. The efficiency scores of the original system corresponding to these directions are 0.891 and 0.935, respectively.

Table 1. DMUs' data (extracted from [3]).

	Input 1	Input 2	Output 1	Output 2
A	150	0.2	14000	3500
B	400	0.7	14000	21000
C	320	1.2	42000	10500
D	520	2.0	28000	42000
E	350	1.2	19000	25000
F	320	0.7	14000	15000

DMU scores for several ranking methods are reported in *Table 2*. The efficiency scores of the new systems after excluding corresponding units are given in the first and second columns of *Table 2*. In both situations, excluding units A and B causes the most significant and minor increase in system-wide efficiency; therefore, they are characterized as the best and worst DMUs among ERM-efficient DMUs, respectively. In confirming our result, all other ranking methods address this point, namely that unit A is the most efficient DMU.

Table 2. DMUs' scores for several ranking methods.

Our Results Using Eq. (9)		Our Results Using Eq. (10)		ERM		CEA		CEB		Benchmarking		AP	
A	0.962	A	0.971	A	1.000	A	0.764	A	1.000	A	1	A	2.000
D	0.947	D	0.972.	B	1.000	B	0.700	D	1.000	B	2	C	1.406
C	0.941	C	0.962	C	1.000	D	0.700	E	0.974	E	3	B	1.400
B	0.926	B	0.960	D	1.000	E	0.696	B	0.955	C	4.5	D	1.130
-	-	-	-	E	0.849	C	0.643	C	0.886	D	4.5	E	0.977
-	-	-	-	F	0.741	F	0.608	F	0.847	-	-	F	0.867

Example 2. Consider seven DMUs, A, B, ..., and G, that use two inputs to produce two outputs, as in *Table 3*. Using these data, we illustrate how our proposed approach can incorporate the DM's preferences information in ranking DMUs. Therefore, this approach can provide a ranking order for DMUs based on both the PPS's information and the given information by DM. In addition, we show the method's strength in ranking nonextreme efficient DMUs.

Table 3. The data set for Example 2.

	I_1	I_2	O_1	O_2
A	3	4	5	3
B	6	3	2	4
C	3	7	3	3
D	10	3	2	1
E	1	5	4	5
F	5	10/3	3	11/3
G	2	9/2	9/2	4

Table 4. The results for example 2 before and after taking into account DM's preferences information.

	ERM	AP	Original	Weighted
A	1.000	1.250	0.551	0.827
B	1.000	1.210	0.600	0.815
C	0.349	0.459	-	-
D	0.252	0.533	-	-
E	1.000	2.500	0.589	0.842
F	1.000	1.000	0.583	0.819
G	1.000	1.000	0.539	0.820

Let the DM's preferences for I_1 I_2 O_1 O_2 and are 1, 0.4, 0.8, and 0.2. According to the *Property (7)* of *Model (6)* $(\xi_1,\xi_2,\psi_1,\psi_2)=(1,2.5,1.25,5)$. Therefore, to incorporate this information into consideration, the direction vector (here we use the direction *Eq. (10)*) used in *Eq. (13)* should be modified as follows:

$$g=(10,7,5,5)\Rightarrow g'=(10,17.5,6.25,25). \quad \textbf{(16)}$$

Table 4 displays the results before and after considering DM's preference information. The results in *Table 4* are exciting. Egregious differences exist between the ranking orders before and after considering DM's preference information. For instance, the order of unit B before and after considering DM's preferences information is one and five, respectively. Therefore, our experiment emphasizes the importance of incorporating DM's preferences and shows that it may change the ranking orders.

Furthermore, with the AP model, nonextreme efficient units, F and G, have the same super-efficiency scores equal to 1, and it fails to differentiate between them. However, the proposed method successfully ranks these units. Therefore, this example demonstrates the superiority of our proposed method in comparison to the super-efficiency technique (AP model) from ranking nonextreme efficient DMUs and accuracy points of view.

5 | Conclusion

In this research, we proposed a new ranking methodology for ranking extreme and nonextreme efficient DMUs based on system-wide performance. In our framework, using the concept of directional distance function, we generalized the approach presented by Cooper et al. [22] and proposed a new SDSBM. The SDSBM has many properties and advantages, e.g., the DM’s preference information can be explicitly considered account. Using SDSBM and based upon the influence of the individual DMUs' performances on the system-wide performance, our ranking method was proposed.

To recapitulate the main idea behind our ranking method: The more influence (increase) on the system-wide performance, the better DMU_a performs. Our ranking method has many desirable properties such as feasibility, ranking both extreme and nonextreme efficient DMUs, and incorporating DM's preferences information into the ranking. The illustrative examples demonstrate the advantages of the proposed method in detail.

Conflict of Interest

The authors declare no conflict of interest.

Data Availability

All data are included in the text.

Funding

This research received no specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

References

[1] Charnes, A., Cooper, W. W., & Rhodes, E. (1978). Measuring the efficiency of decision making units. *European journal of operational research*, *2*(6), 429–444. https://doi.org/10.1016/0377-2217(78)90138-8

[2] Sexton, T. R., Silkman, R. H., & Hogan, A. J. (1986). Data envelopment analysis: Critique and extensions. *New directions for program evaluation*, *1986*(32), 73–105. https://doi.org/10.1002/ev.1441

[3] Adler, N., Friedman, L., & Sinuany-Stern, Z. (2002). Review of ranking methods in the data envelopment analysis context. *European journal of operational research*, *140*(2), 249–265. https://doi.org/10.1016/S0377-2217(02)00068-1

[4] Cook, W. D., & Seiford, L. M. (2009). Data envelopment analysis (DEA)-thirty years on. *European journal of operational research*, *192*(1), 1–17. https://doi.org/10.1016/j.ejor.2008.01.032

[5] Andersen, P., & Petersen, N. C. (1993). A procedure for ranking efficient units in data envelopment analysis. *Management science*, *39*(10), 1261–1264. https://doi.org/10.1287/mnsc.39.10.1261

[6] Torgersen, A. M., Førsund, F. R., & Kittelsen, S. A. C. (1996). Slack-adjusted efficiency measures and ranking of efficient units. *Journal of productivity analysis*, *7*, 379–398. https://doi.org/10.1007/BF00162048

[7] Jahanshahloo, G. R., Lotfi, F. H., Shoja, N., Tohidi, G., & Razavyan, S. (2004). Ranking using l1-norm in data envelopment analysis. *Applied mathematics and computation*, *153*(1), 215–224. https://doi.org/10.1016/S0096-3003(03)00625-8

[8] Balf, F. R., Rezai, H. Z., Jahanshahloo, G. R., & Lotfi, F. H. (2012). Ranking efficient DMUs using the Tchebycheff norm. *Applied mathematical modelling*, *36*(1), 46–56. https://doi.org/10.1016/j.apm.2010.11.077

[9] Amirteimoori, A., Jahanshahloo, G., & Kordrostami, S. (2005). Ranking of decision making units in data envelopment analysis: A distance-based approach. *Applied mathematics and computation*, *171*(1), 122–135. https://doi.org/10.1016/j.amc.2005.01.065

[10] Troutt, M. D. (1995). A maximum decisional efficiency estimation principle. *Management science*, *41*(1), 76–82. https://doi.org/10.1287/mnsc.41.1.76

[11] Ganley, J. A., & Cubbin, J. S. (1992). *Public sector efficiency measurement: Applications of data envelopment analysis*. Elsevier Science Inc. https://dl.acm.org/doi/10.5555/573173

[12] Hosseinzadeh Lotfi, F., Jahanshahloo, G. R., & Memariani, A. (2000). A method for finding common set of weights by multiple objective programming in data envelopment analysis. *Southwest journal of pure and applied mathematics [electronic only]*, *2000*(1), 44–54. https://eudml.org/doc/222875

[13] Liu, F. H. F., & Peng, H. H. (2008). Ranking of units on the DEA frontier with common weights. *Computers & operations research*, *35*(5), 1624–1637. https://doi.org/10.1016/j.cor.2006.09.006

[14] Jahanshahloo, G. R., Lotfi, F. H., Rezai, H. Z., & Balf, F. R. (2005). Using Monte Carlo method for ranking efficient DMUs. *Applied mathematics and computation*, *162*(1), 371–379. https://doi.org/10.1016/j.amc.2003.12.139

[15] Lotfi, F. H., Fallahnejad, R., & Navidi, N. (2011). Ranking efficient units in DEA by using TOPSIS method. *Applied mathematical sciences*, *5*(17), 805–815. https://B2n.ir/mh1784

[16] Luenberger, D. G. (1992). Benefit functions and duality. *Journal of mathematical economics*, *21*(5), 461–481. https://doi.org/10.1016/0304-4068(92)90035-6

[17] Luenberger, D. G. (1995). *Microeconomic theory McGraw-Hill*. , Inc., New York. https://B2n.ir/nk5701

[18] Chambers, R. G., Chung, Y., & Färe, R. (1996). Benefit and distance functions. *Journal of economic theory*, *70*(2), 407–419. https://doi.org/10.1006/jeth.1996.0096

[19] Chambers, R. G., Chung, Y., & Färe, R. (1998). Profit, directional distance functions, and Nerlovian efficiency. *Journal of optimization theory and applications*, *98*, 351–364. https://doi.org/10.1023/A:1022637501082

[20] Pastor, J. T., Ruiz, J. L., & Sirvent, I. (1999). An enhanced DEA Russell graph efficiency measure. *European journal of operational research*, *115*(3), 596–607. https://doi.org/10.1016/S0377-2217(98)00098-8

[21] Tone, K. (2001). A slacks-based measure of efficiency in data envelopment analysis. *European journal of operational research*, *130*(3), 498–509. https://doi.org/10.1016/S0377-2217(99)00407-5

[22] Cooper, W. W., Seiford, L. M., Tone, K. (2007). *Data envelopment analysis: A comprehensive text with models, applications, references and DEA-solver software* (Vol. 2). Springer. https://doi.org/10.1007/978-0-387-45283-8

[23] Cooper, W. W., Huang, Z., Li, S. X., Parker, B. R., & Pastor, J. T. (2007). Efficiency aggregation with enhanced Russell measures in data envelopment analysis. *Socio-economic planning sciences*, *41*(1), 1–21. https://doi.org/10.1016/j.seps.2006.03.001

[24] Shephard, R. W. (1971). *Theory of cost and production functions*. Princeton University Press. https://B2n.ir/dr2675

[25] Cooper, A. C. W. W., & Charnes, W. (1962). Programming with linear fractional functionals. *Naval research logistics quarterly*, *9*(3), 181–186. https://b2n.ir/wg5765