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MultiType/MultiRung Orthopair/Multilinguistic Fuzzy Set with Applications

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Abstract

A fuzzy set maps each element to a degree in $[0, 1]$, representing partial membership and enabling reasoning and operations under vagueness. Type-n fuzzy sets nest memberships recursively: each value is a fuzzy set of order $n-1$, capturing higher-order uncertainty and footprint variability. A q-rung orthopair fuzzy set assigns membership and nonmembership degrees satisfying $\mu^q + \nu^q \leq 1$, modeling hesitation and generalizing intuitionistic and Pythagorean forms. A linguistic fuzzy set links verbal terms (with hedges) to membership functions over data, enabling interpretable, human-centric reasoning with calibrated semantics. In this paper, we study the MultiType, MultiRung Orthopair, and MultiLinguistic Fuzzy Sets, which extend the type, q-rung orthopair, and linguistic frameworks into multi-structured forms.

Keywords: Fuzzy set, Neutrosophic set, Type-n fuzzy set, Q-rung orthopair fuzzy set, Linguistic fuzzy set.

1 | Introduction

We collect the basic terminology and notation used in what follows.

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1.1 | Type-n Fuzzy Sets

Type-n fuzzy sets nest memberships recursively: each value is a fuzzy set of order $n-1$, thereby capturing higher-order uncertainty and footprint variability. Related notions include the Type-2 Fuzzy Set[1, 2, 3], the Type-3 Fuzzy Set[4, 5], and the Type-2 Neutrosophic Set[6, 7]. The formal definitions of Fuzzy Sets and Type-n Fuzzy Sets are provided below.

Definition 1 (Fuzzy Set)**.** [8, 9] A *fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \to [0,1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a *fuzzy relation on* τ if

$$\delta(y,z) \le \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Definition 2 (Type-n Fuzzy Sets)**.** [10, 11] Let X be a universal set and $n > 1$. A *Type-n Fuzzy Set* $\tilde{A}$ is defined as:

$$\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x)\rangle \mid x \in X\},$$

where the membership function $\mu_{\tilde{A}}(x)$ is defined recursively as:

$$\mu_{\tilde{A}}(x) : X \to M_{n-1}[0,1],$$

and $M_{n-1}[0,1]$ denotes the set of Type-$(n-1)$ fuzzy sets over the unit interval $[0,1]$.

Special Cases:

- *Type-1 Fuzzy Set:* $\mu_{\tilde{A}}(x) \in [0,1]$.
- *Type-2 Fuzzy Set:* $\mu_{\tilde{A}}(x) = \{\langle u, \mu_{\tilde{A},x}(u)\rangle \mid u \in [0,1]\}$.
- *Type-3 Fuzzy Set:* $\mu_{\tilde{A}}(x) = \{\langle u, \{\langle v, \mu_{\tilde{A},x,u}(v)\rangle \mid v \in [0,1]\}\rangle \mid u \in [0,1]\}$.

General Case: For any $n \ge 1$,

$$\mu_{\tilde{A}}(x) \in M_{n-1}[0,1],$$

where the recursion terminates when $n = 1$.

Example 1 (Type-2 fuzzy set for "Warm" temperature (instantiates the Type-n definition with $n = 2$))**.** Let the universe be temperature $X = \mathbb{R}$ (in °C). We model the linguistic concept $\widetilde{A} =$ "Warm" as an *interval Type-2* fuzzy set to encode sensor/semantic uncertainty. Define an *upper* and a *lower* Type-1 membership (triangular shapes)

$$\overline{\mu}(x) := \max\Big\{0, \min\Big(\tfrac{x-18}{25-18}, \tfrac{32-x}{32-25}\Big)\Big\}, \qquad \underline{\mu}(x) := \max\Big\{0, \min\Big(\tfrac{x-20}{25-20}, \tfrac{30-x}{30-25}\Big)\Big\}.$$

For each $x \in X$, the secondary membership is the unit-height interval on $[0,1]$,

$$\mu_{\widetilde{A},x}(u) \;=\; \begin{cases} 1, & u \in [\underline{\mu}(x), \overline{\mu}(x)], \\ 0, & \text{otherwise}, \end{cases}$$

so the Type-2 membership is $\mu_{\widetilde{A}}(x) = \{\, \langle u, \mu_{\widetilde{A},x}(u)\rangle : u \in [0,1]\,\}$. This realizes the recursive clause of the Type-n definition with $n = 2$ (each primary grade is itself a fuzzy set over $[0,1]$). For instance, at $x = 24$, $\underline{\mu}(24) = \min(\frac{24-20}{5}, \frac{30-24}{5}) = 0.8$ and $\overline{\mu}(24) = \min(\frac{24-18}{7}, \frac{32-24}{7}) = \frac{6}{7} \approx 0.857$, hence the footprint-of-uncertainty is [0.8, 0.857].

1.2 | Q-Rung Orthopair Fuzzy Set

A (finite) q-rung orthopair fuzzy set (q-ROFS) models uncertainty by assigning to each element two grades—membership μ and non-membership ν—with the constraint

$$\mu^q + \nu^q \;\le\; 1,$$

which yields a flexible representation encompassing several earlier frameworks [12, 13, 14]. A closely related generalization, the q-rung orthopair neutrosophic set, has also been proposed and investigated [15]. The formal definition follows.

Definition 3 (q-Rung Orthopair Fuzzy Set [12])**.** Let X be a (finite) universe. A (finite) q-rung orthopair fuzzy set R on X is specified by two functions $\mu_R, \nu_R : X \to [0,1]$ and is represented as

$$R = \{\langle x, \mu_R(x), \nu_R(x)\rangle \mid x \in X\},$$

subject to

$$0 \;\le\; \big(\mu_R(x)\big)^q + \big(\nu_R(x)\big)^q \;\le\; 1 \quad \text{for all } x \in X, \quad q \ge 1.$$

Example 2 (q-Rung Orthopair Fuzzy Set ($q = 3$) for triage likelihood)**.** Let $X = \{p_1, p_2, p_3\}$ be three patients and $\mathcal{R}$ denote the q-ROFS "requires urgent care". Assign membership μ and nonmembership ν with the q-orthopair constraint $\mu^q + \nu^q \le 1$ for $q = 3$:

x	$\mu(x)$	$\nu(x)$	$\mu(x)^3 + \nu(x)^3$
p_1	0.80	0.40	$0.512 + 0.064 = 0.576\ (\le 1)$
p_2	0.60	0.70	$0.216 + 0.343 = 0.559\ (\le 1)$
p_3	0.30	0.90	$0.027 + 0.729 = 0.756\ (\le 1)$

All entries satisfy the constraint, so $\mathcal{R} = \{\langle x, \mu(x), \nu(x)\rangle : x \in X\}$ is a valid 3-rung orthopair fuzzy set. The hesitation (indeterminacy) per patient is $\pi(x) = (1 - \mu(x)^3 - \nu(x)^3)^{1/3}$, e.g. $\pi(p_1) = (1 - 0.576)^{1/3} \approx 0.76$, quantifying residual uncertainty after specifying μ, ν.

1.3 | Linguistic Fuzzy Set

A linguistic fuzzy set links verbal terms (with hedges) to membership functions over data, enabling interpretable, human-centric reasoning with calibrated semantics (cf.[16, 17, 18, 19]).

Definition 4 (Linguistic domain and semantics)**.** Let X be a nonempty universe. A *linguistic domain* is a triple

$$\mathbb{L} = (S, \preceq, H),$$

where S is a finite (or countable) set of primary linguistic terms (e.g., *low*, *medium*, *high*), $\preceq$ is a total (or partial) order on S, and H is a (finite) set of unary *linguistic hedges* (e.g., *very*, *more or less*) acting on S. Let S_H denote the closure of S under H. A *linguistic semantics* on X is a map

$$\Sigma : \; S_H \longrightarrow \mathsf{F}(X), \qquad \ell \longmapsto \Sigma(\ell) = \mu_\ell(\cdot),$$

where $\mathsf{F}(X) = \{\, \mu : X \to [0,1] \,\}$ is the class of Type-1 fuzzy subsets of X. Thus, each (hedged) term $\ell \in S_H$ is interpreted as a fuzzy subset $\mu_\ell : X \to [0,1]$ of X (with the usual monotonicity of hedges, when imposed).

Example 3 (Linguistic domain and semantics on temperature)**.** Let the universe be temperature $X = \mathbb{R}$ (in °C). Define the linguistic domain

$$\mathbb{L} = (S, \preceq, H), \qquad S = \{\text{cold}, \text{warm}, \text{hot}\}, \quad \text{cold} \prec \text{warm} \prec \text{hot}, \quad H = \{\text{very}, \text{more\_or\_less}\}.$$

Let S_H be the hedge-closure of S under H. The semantics $\Sigma : S_H \to \mathsf{F}(X)$ is given by:

$$\mu_{\text{cold}}(x) = \begin{cases} 1, & x \le 10, \\ \dfrac{20 - x}{10}, & 10 < x < 20, \\ 0, & x \ge 20, \end{cases} \quad \mu_{\text{warm}}(x) = \max\Big\{0,\ \min\Big(\frac{x - 18}{7}, \frac{32 - x}{7}\Big)\Big\},$$

$$\mu_{\text{hot}}(x) = \begin{cases} 0, & x \le 28, \\ \dfrac{x - 28}{12}, & 28 < x < 40, \\ 1, & x \ge 40, \end{cases} \quad \mu_{\text{very}\,\ell}(x) := \big(\mu_\ell(x)\big)^2, \qquad \mu_{\text{more\_or\_less}\,\ell}(x) := \sqrt{\mu_\ell(x)}.$$

For instance, at $x = 26$ we have $\mu_{\text{warm}}(26) = \min\big(\frac{26-18}{7}, \frac{32-26}{7}\big) = \frac{6}{7}$, so $\mu_{\text{very warm}}(26) = \big(\frac{6}{7}\big)^2 = \frac{36}{49} \approx 0.735$.

Definition 5 (Linguistic Fuzzy Set (LFS))**.** Given a linguistic domain $\mathbb{L} = (S, \preceq, H)$ with semantics Σ, a *linguistic fuzzy set* on X is the family

$$\widetilde{A} \;=\; \Big\{ \big\langle x,\ \{\mu_\ell(x)\}_{\ell \in S_H} \big\rangle \;:\; x \in X \Big\},$$

equivalently the assignment $x \mapsto \{\mu_\ell(x)\}_{\ell \in S_H}$ of degrees expressing how appropriate each linguistic label ℓ is for x. For any fixed ℓ, the *ℓ-cut* of $\widetilde{A}$ is the fuzzy set $\Sigma(\ell) = \mu_\ell(\cdot)$ on X. *Aggregation-to-score* (when a scalar summary is needed) can be effected by a monotone operator Agg:

$$\mu^*(x) \;:=\; \mathrm{Agg}\big(\{\mu_\ell(x)\}_{\ell \in S_H}\big) \;\in [0,1].$$

Example 4 (Linguistic fuzzy set over a finite temperature sample)**.** Let $X = \{19,\ 26,\ 35\} \subset \mathbb{R}$ (in °C) and use the semantics from the previous example. The LFS $\widetilde{A} = \{\langle x, \{\mu_\ell(x)\}_{\ell \in S_H} \rangle : x \in X\}$ assigns, e.g., for $\ell \in \{\text{cold}, \text{warm}, \text{hot}, \text{very warm}\}$:

x	$\mu_{\text{cold}}(x)$	$\mu_{\text{warm}}(x)$	$\mu_{\text{hot}}(x)$	$\mu_{\text{very warm}}(x)$
19	0.10	$\frac{1}{7} \approx 0.143$	0	$(\frac{1}{7})^2 \approx 0.020$
26	0	$\frac{6}{7} \approx 0.857$	0	$(\frac{6}{7})^2 \approx 0.735$
35	0	0	$\frac{7}{12} \approx 0.583$	0

A scalar summary may be obtained by a monotone aggregation, e.g. the equal-weight mean over $\{\text{cold}, \text{warm}, \text{hot}\}$:

$$\mu^*(x) = \tfrac{1}{3}\big(\mu_{\text{cold}}(x) + \mu_{\text{warm}}(x) + \mu_{\text{hot}}(x)\big),$$

yielding $\mu^*(19) \approx 0.081,\ \mu^*(26) \approx 0.286,\ \mu^*(35) \approx 0.194.$

2 | Result: MultiType Fuzzy Set

A MultiType Fuzzy Set unifies different fuzzy set types into one structure to represent diverse layers of uncertainty.

Definition 6 (Type hierarchy and notation)**.** Let X be a nonempty universe. For $n \in \mathbb{N}_{\geq 1}$, denote by $\mathsf{T}(n)$ the class of *Type-n* fuzzy memberships on X, defined recursively by

$$\mathsf{T}(1) := \big\{\mu : X \to [0,1]\big\}, \qquad \mathsf{T}(n) := \big\{\mu : X \to \mathsf{T}(n-1)\big\}\ (n \geq 2).$$

Thus Type-2 memberships assign to each $x \in X$ a (Type-1) fuzzy set on $[0,1]$ (i.e., a secondary membership), Type-3 assigns a fuzzy set of fuzzy sets, and so on.

Definition 7 (MultiType Fuzzy Set (MTFS))**.** Let J be a finite index set of *types*. To each $j \in J$ attach a positive integer $n_j \geq 1$ (the order of the type) and optionally a nonnegative weight w_j with $\sum_{j \in J} w_j = 1$. A *MultiType Fuzzy Set* on X is the family

$$\widetilde{\mathcal{A}} \;:=\; \Big\{ \langle x,\ (\mu^{(j)}(x))_{j \in J} \rangle \;:\; x \in X \Big\}, \qquad \mu^{(j)} \in \mathsf{T}(n_j) \text{ for each } j \in J.$$

Equivalently, it is a map $x \mapsto (\mu^{(j)}(x))_{j \in J}$ with the j-th coordinate a Type-n_j membership. For j fixed, the *j-projection* of $\widetilde{\mathcal{A}}$ is the Type-n_j fuzzy set $\widetilde{A}^{(j)} = \{\langle x, \mu^{(j)}(x) \rangle : x \in X\}$.

Remark 1 (Reductions and scalarization)**.** (i) If $|J| = 1$ and $n_j = 1$, an MTFS is an ordinary (Type-1) fuzzy set. If $|J| = 1$ and $n_j = 2$, it is a Type-2 fuzzy set, etc.

(ii) For decision or ranking purposes, one often uses a *scalarization* that maps each Type-n_j coordinate to a representative in $[0,1]$. Let $\rho_j : \mathsf{T}(n_j) \to [0,1]$ be a *type-specific reduction* (e.g., height or centroid for Type-1; centroid-of-set or KM-type reduction for interval Type-2). Define the aggregated grade

$$\mu^*(x) \;:=\; \mathrm{Agg}\big(\, w_j\, \rho_j(\mu^{(j)}(x)) \;:\; j \in J \,\big),$$

where Agg is any monotone averaging operator (e.g., weighted arithmetic mean or OWA). This yields a single Type-1 summary $\mu^* : X \to [0,1]$ when needed.

Example 3 (MTFS with Type-1 and interval Type-2 coordinates)**.** Let $X = \{A, B, C\}$ be three candidates. Take $J = \{\text{experience}, \text{ skill}\}$ with $n_{\text{experience}} = 1$ and $n_{\text{skill}} = 2$. Define:

$$\mu^{(\text{experience})}(A) = 0.7,\ \mu^{(\text{experience})}(B) = 0.5,\ \mu^{(\text{experience})}(C) = 0.4.$$

For the Type-2 skill coordinate, use interval Type-2 footprints of uncertainty (FOU): for $x \in X$, the secondary membership is 1 on $[L_x, U_x] \subseteq [0, 1]$ and 0 otherwise:

$$[L_A, U_A] = [0.6, 0.8], \qquad [L_B, U_B] = [0.4, 0.7], \qquad [L_C, U_C] = [0.9, 1.0].$$

Choose reductions $\rho_{\text{experience}}(\mu) = \mu$ and $\rho_{\text{skill}}(\text{IT2}) = (L_x + U_x)/2$ (midpoint of the FOU), and weights $w_{\text{experience}} = w_{\text{skill}} = \frac{1}{2}$. The aggregated scores are

$$\mu^*(A) = \tfrac{1}{2} \cdot 0.7 + \tfrac{1}{2} \cdot \tfrac{0.6+0.8}{2} = 0.7,$$
$$\mu^*(B) = \tfrac{1}{2} \cdot 0.5 + \tfrac{1}{2} \cdot \tfrac{0.4+0.7}{2} = 0.525,$$
$$\mu^*(C) = \tfrac{1}{2} \cdot 0.4 + \tfrac{1}{2} \cdot \tfrac{0.9+1.0}{2} = 0.675.$$

Hence A ranks highest (0.70), followed by C (0.675) and B (0.525). The full MTFS representation still retains the richer Type-2 skill uncertainty for downstream reasoning.

Example 4 (Air-quality assessment with heterogeneous uncertainty layers)**.** Let $X = \{t_1, t_2, t_3\}$ denote time points (9:00, 12:00, 18:00). We assess the concept "*unhealthy air*" using two types: $J = \{j_1, j_2\}$ with $n_{j_1} = 1$ (Type-1) and $n_{j_2} = 2$ (Type-2, interval).

Type-1 coordinate (calibrated $PM_{2.5}$): Define $\mu^{(j_1)} : X \to [0, 1]$ by a trapezoid on the measured $\text{PM}_{2.5}$ concentration (0 at ≤ 35, 1 at ≥ 100, linear in between). Suppose the measured values yield

$$\mu^{(j_1)}(t_1) = 0.30, \qquad \mu^{(j_1)}(t_2) = 0.65, \qquad \mu^{(j_1)}(t_3) = 0.90.$$

Type-2 coordinate (crowdsourced odor/haze, interval Type-2): For each $t \in X$, let the secondary membership be the interval footprint $[\underline{\mu}(t), \overline{\mu}(t)]$ (all secondary grades in the band have membership 1):

$$[\underline{\mu}, \overline{\mu}](t_1) = [0.20, 0.50], \quad [\underline{\mu}, \overline{\mu}](t_2) = [0.55, 0.80], \quad [\underline{\mu}, \overline{\mu}](t_3) = [0.70, 0.95].$$

Thus the MTFS at each t is $\big(\mu^{(j_1)}(t),\ \text{interval } [\,\underline{\mu}(t), \overline{\mu}(t)\,]\big)$. For ranking, pick a type-specific reduction ρ_{j_2} as the interval midpoint, and aggregate with weights $w_{j_1} = w_{j_2} = 1/2$:

$$\mu^*(t) = \tfrac{1}{2}\, \mu^{(j_1)}(t) + \tfrac{1}{2}\, \tfrac{\underline{\mu}(t)+\overline{\mu}(t)}{2},$$

giving, e.g., $\mu^*(t_1) = 0.25,\ \mu^*(t_2) = 0.675,\ \mu^*(t_3) = 0.825$. The MTFS thus fuses a precise sensor (Type-1) with an uncertain crowd signal (Type-2) without forcing both into a single type.

Example 5 (Maintenance scheduling with multi–type evidence)**.** Consider three machines $X = \{M_1, M_2, M_3\}$ and the concept "*needs preventive maintenance this week*". Use $J = \{j_1, j_2, j_3\}$ with orders $n_{j_1} = 1$ (Type-1), $n_{j_2} = 2$ (interval Type-2), and $n_{j_3} = 3$ (Type-3) to encode heterogeneous evidence.

Type-1 (hard counters): $\mu^{(j_1)}(M_i)$ derived from vibration RMS vs. threshold (normalized to $[0, 1]$):

$$\mu^{(j_1)}(M_1) = 0.40, \quad \mu^{(j_1)}(M_2) = 0.75, \quad \mu^{(j_1)}(M_3) = 0.50.$$

Type-2 (interval, operator reports): Each M_i has an interval footprint from subjective noise/heat reports:

$$[\underline{\mu}, \overline{\mu}](M_1) = [0.30, 0.55], \quad [\underline{\mu}, \overline{\mu}](M_2) = [0.60, 0.90], \quad [\underline{\mu}, \overline{\mu}](M_3) = [0.45, 0.70].$$

Type-3 (model-ensemble meta-uncertainty): For each M_i, define a Type-3 object as a *band of admissible Type-2 footprints* representing alternative digital-twin models; formally,

$$\mu^{(j_3)}(M_i) \ :=\ \big\{\, \text{interval Type-2 footprints } [\ell, u] \text{ with } \ell \in [L_i^-, L_i^+],\ u \in [U_i^-, U_i^+],\ \ell \leq u \,\big\},$$

with tertiary membership 1 for all such inner Type-2s. Take

$$(L_1^-, L_1^+; U_1^-, U_1^+) = (0.25, 0.35; 0.55, 0.65),$$
$$(L_2^-, L_2^+; U_2^-, U_2^+) = (0.55, 0.65; 0.85, 0.95),$$

$$(L_3^-, L_3^+; U_3^-, U_3^+) = (0.40, 0.50; 0.70, 0.80).$$

A simple reduction ρ_{j_3} maps this Type-3 band to the midpoint of admissible midpoints: $\rho_{j_3}(M_i) = \frac{1}{2}\big(\frac{L_i^- + L_i^+}{2} + \frac{U_i^- + U_i^+}{2}\big)$, yielding $\rho_{j_3}(M_1) = 0.45,\ \rho_{j_3}(M_2) = 0.85,\ \rho_{j_3}(M_3) = 0.60$.

Aggregation. With weights $(w_{j_1}, w_{j_2}, w_{j_3}) = (0.3, 0.3, 0.4)$ and ρ_{j_2} the interval midpoint,

$$\mu^*(M_i) = 0.3\,\mu^{(j_1)}(M_i) + 0.3\,\frac{\underline{\mu}(M_i) + \overline{\mu}(M_i)}{2} + 0.4\,\rho_{j_3}(M_i),$$

so that, numerically,

$$\mu^*(M_1) = 0.405, \qquad \mu^*(M_2) = 0.815, \qquad \mu^*(M_3) = 0.565.$$

Hence the MTFS ranks $M_2 > M_3 > M_1$ for preventive action, coherently combining precise sensors, interval–valued human inputs, and model–ensemble meta-uncertainty in a single structure that no single Type-n fuzzy set could natively capture.

Theorem 1 (MultiType generalizes Type-n). *Fix a nonempty universe X. For every $n \in \mathbb{N}_{\geq 1}$, the class $\mathsf{T}(n)$ of Type-n fuzzy memberships embeds (indeed, is isomorphic) to a subclass of MultiType Fuzzy Sets (MTFS). In particular, if $J = \{j_\star\}$ and $n_{j_\star} = n$, then*

$$\Phi_n :\ \mathsf{T}(n) \longrightarrow \mathrm{MTFS}(X; J, \{n_{j_\star}\}), \qquad \Phi_n(\mu) \ :=\ \big\{ \langle x,\ (\mu^{(j_\star)}(x))\rangle :\ \mu^{(j_\star)}(x) := \mu(x) \big\}$$

is a bijection onto the single–type subclass $\{\widetilde{\mathcal{A}} :\ |J| = 1,\ n_{j_\star} = n\}$. Consequently, MTFS strictly generalizes Type-n, because it also admits $|J| > 1$ with (possibly) heterogeneous orders $\{n_j\}_{j \in J}$ that cannot, in general, be represented as any single Type-m.

Proof: (*Embedding/isomorphism*) Fix n and $J = \{j_\star\}$ with $n_{j_\star} = n$. By Definition of MTFS, any $\widetilde{\mathcal{A}}$ with $|J| = 1$ is exactly a map $x \mapsto \mu^{(j_\star)}(x) \in \mathsf{T}(n)$, i.e., a Type-$n$ fuzzy set. Thus Φ_n above is well defined and invertible via

$$\Phi_n^{-1}\big(\widetilde{\mathcal{A}}\big)(x) \ :=\ \mu^{(j_\star)}(x),$$

showing a one-to-one correspondence between $\mathsf{T}(n)$ and the single–type MTFS subclass.

(*Strictness*) Choose $J = \{j_1, j_2\}$ with $(n_{j_1}, n_{j_2}) = (1, 2)$. An MTFS assigns to each $x \in X$ a pair $\big(\mu^{(j_1)}(x), \mu^{(j_2)}(x)\big) \in \mathsf{T}(1) \times \mathsf{T}(2)$. Suppose, towards contradiction, that there exists some m and a representation of *every* such MTFS as a single Type-m membership $\widehat{\mu}(x) \in \mathsf{T}(m)$ that preserves, for all x, both the Type-1 and Type-2 coordinates as recoverable invariants (i.e., there exist fixed operators π_1, π_2 with $\pi_1(\widehat{\mu}(x)) = \mu^{(j_1)}(x)$ and $\pi_2(\widehat{\mu}(x)) = \mu^{(j_2)}(x)$). But $\mathsf{T}(1)$ and $\mathsf{T}(2)$ have, in general, non-isomorphic signatures (a Type-2 object is a fuzzy set of (Type-1) grades on $[0, 1]$, while a Type-1 object is a single grade); no single Type-m object can *universally* encode both *and* guarantee fixed, lossless projections π_1, π_2 for arbitrary choices of the two coordinates. Hence such a reduction does not exist in general, and MTFS with heterogeneous orders cannot be captured by any single Type-m class. Therefore MTFS strictly extends all Type-n classes. □

3| Result: MultiRung Orthopair Fuzzy Set

A MultiRung Orthopair Fuzzy Set extends q-rung fuzzy sets by allowing multiple rungs for richer uncertainty modeling.

Definition 8 (MultiRung Orthopair Fuzzy Set (MROFS)). Let X be a nonempty universe and let $R = \{q_1, \ldots, q_m\} \subseteq [1, \infty)$ be a finite set of *rungs*. A *MultiRung Orthopair Fuzzy Set* on X assigns, to each $x \in X$ and each rung $q \in R$, a membership degree $\mu_q(x) \in [0, 1]$ and a nonmembership degree $\nu_q(x) \in [0, 1]$ such that the q-orthopair constraint holds:

$$\big(\mu_q(x)\big)^q + \big(\nu_q(x)\big)^q \ \leq\ 1 \qquad (\forall\, x \in X,\ \forall\, q \in R).$$

We write

$$\mathcal{A} \ =\ \Big\{ \langle x,\ \{(\mu_q(x), \nu_q(x))\}_{q \in R}\rangle \ :\ x \in X \Big\}.$$

When $|R| = 1$ with $R = \{q\}$, this reduces to a standard q-rung orthopair fuzzy set.

Notation 1 (Hesitation and scoring across rungs)**.** *For each rung $q \in R$ and $x \in X$, define the q-*hesitation *(indeterminacy) by*

$$\pi_q(x) \;:=\; \big(1-\mu_q(x)^q-\nu_q(x)^q\big)^{1/q} \;\in [0,1].$$

To compare elements, one may use aggregated indices, e.g.,

$$S(x) \;:=\; \frac{1}{|R|}\sum_{q\in R}\Big(\mu_q(x)^q-\nu_q(x)^q\Big) \quad \textit{(score)}, \qquad H(x) \;:=\; \frac{1}{|R|}\sum_{q\in R}\pi_q(x) \quad \textit{(average hesitation)}.$$

Larger $S(x)$ is preferable; $H(x)$ quantifies residual uncertainty.

Definition 9 (Basic set operations (rungwise))**.** Let $\mathcal{A},\mathcal{B}$ be MROFS on the same (X,R). Their *union*, *intersection*, and *complement* are defined for each $q \in R$ and $x \in X$ by

$$\begin{aligned}
(\mu_q^{\mathcal{A}\cup\mathcal{B}}(x),\ \nu_q^{\mathcal{A}\cup\mathcal{B}}(x)) &:= \big(\max\{\mu_q^{\mathcal{A}}(x),\mu_q^{\mathcal{B}}(x)\},\ \min\{\nu_q^{\mathcal{A}}(x),\nu_q^{\mathcal{B}}(x)\}\big),\\
(\mu_q^{\mathcal{A}\cap\mathcal{B}}(x),\ \nu_q^{\mathcal{A}\cap\mathcal{B}}(x)) &:= \big(\min\{\mu_q^{\mathcal{A}}(x),\mu_q^{\mathcal{B}}(x)\},\ \max\{\nu_q^{\mathcal{A}}(x),\nu_q^{\mathcal{B}}(x)\}\big),\\
(\mu_q^{\mathcal{A}^c}(x),\ \nu_q^{\mathcal{A}^c}(x)) &:= \big(\nu_q^{\mathcal{A}}(x),\ \mu_q^{\mathcal{A}}(x)\big).
\end{aligned}$$

Each operation preserves the rungwise constraint $\mu_q^q+\nu_q^q \le 1$.

Example 8 (Two-rung (Pythagorean and $q{=}3$) orthopair evaluation)**.** Let $X=\{x_1,x_2,x_3\}$ and $R=\{2,3\}$. Specify, for all $x \in X$,

	$\mu_2(x)$	$\nu_2(x)$	$\mu_3(x)$	$\nu_3(x)$
x_1	0.70	0.50	0.62	0.60
x_2	0.55	0.35	0.40	0.80
x_3	0.30	0.80	0.50	0.70

Each row satisfies $\mu_2^2+\nu_2^2 \le 1$ and $\mu_3^3+\nu_3^3 \le 1$. Compute scores and hesitations:

Scores.

$$\begin{aligned}
S(x_1) &= \tfrac{1}{2}\big(0.70^2-0.50^2+0.62^3-0.60^3\big) = \tfrac{1}{2}(0.24+0.022) \approx 0.131,\\
S(x_2) &= \tfrac{1}{2}\big(0.55^2-0.35^2+0.40^3-0.80^3\big) = \tfrac{1}{2}(0.180-0.448) \approx -0.134,\\
S(x_3) &= \tfrac{1}{2}\big(0.30^2-0.80^2+0.50^3-0.70^3\big) = \tfrac{1}{2}(-0.550-0.218) \approx -0.384.
\end{aligned}$$

Hence x_1 ranks highest by score.

Average hesitations.

$$\begin{aligned}
H(x_1) &= \tfrac{1}{2}\Big((1-0.70^2-0.50^2)^{1/2}+(1-0.62^3-0.60^3)^{1/3}\Big) \approx \tfrac{1}{2}(0.510+0.817) = 0.664,\\
H(x_2) &\approx \tfrac{1}{2}\big(\sqrt{0.575}+\sqrt[3]{0.424}\big) \approx \tfrac{1}{2}(0.758+0.753) = 0.756,\\
H(x_3) &\approx \tfrac{1}{2}\big(\sqrt{0.270}+\sqrt[3]{0.532}\big) \approx \tfrac{1}{2}(0.520+0.810) = 0.665.
\end{aligned}$$

Element x_1 is best (largest S) with moderate hesitation, while x_2 and x_3 are less preferred and exhibit higher or comparable uncertainty.

Example 9 (Hospital triage across conservativeness rungs)**.** Let $X=\{\mathrm{P}_1,\mathrm{P}_2,\mathrm{P}_3\}$ be patients to be classified under the concept "*requires ICU now*". Choose two rungs $R=\{2,5\}$, where $q=2$ encodes a conservative clinical stance (Pythagorean constraint) and $q=5$ a more permissive stance (greater tolerance to simultaneous membership/nonmembership evidence). Assign (μ_q,ν_q) as follows:

	$q=2$		$q=5$	
patient	μ_2	ν_2	μ_5	ν_5
P_1	0.80	0.50	0.85	0.45
P_2	0.60	0.60	0.65	0.55
P_3	0.30	0.90	0.40	0.90

Each pair satisfies the rungwise orthopair constraint:

$$\begin{aligned}
\mathrm{P}_1: &\ 0.80^2+0.50^2=0.89\le 1, \quad 0.85^5+0.45^5\approx 0.462\le 1;\\
\mathrm{P}_2: &\ 0.60^2+0.60^2=0.72\le 1, \quad 0.65^5+0.55^5\approx 0.166\le 1;\\
\mathrm{P}_3: &\ 0.30^2+0.90^2=0.90\le 1, \quad 0.40^5+0.90^5\approx 0.601\le 1.
\end{aligned}$$

Clinically, μ_q summarizes evidence for immediate ICU need (vital signs, labs), while ν_q summarizes counter-evidence (stability, response to oxygen). A rung-averaged score such as $S(x)=\frac{1}{|R|}\sum_{q\in R}(\mu_q(x)^q-\nu_q(x)^q)$ orders the patients as $S(\mathrm{P}_1)>S(\mathrm{P}_2)>S(\mathrm{P}_3)$, supporting priority P_1 (strongest ICU indication), then P_2, with P_3 least urgent under both conservative and permissive stances.

Example 10 (Loan approval under multiple risk postures)**.** Let $X=\{\mathrm{A},\mathrm{B},\mathrm{C}\}$ be loan applicants to be assessed under the concept "*creditworthy*". Take three rungs $R=\{2,3,5\}$, reflecting risk postures from conservative (q=2) to permissive (q=5). For each q, μ_q aggregates supportive factors (income stability, low DTI, good bureau score) and ν_q aggregates adverse factors (delinquencies, thin file). Set

	$q=2$		$q=3$		$q=5$	
applicant	μ_2	ν_2	μ_3	ν_3	μ_5	ν_5
A	0.75	0.45	0.80	0.40	0.85	0.35
B	0.55	0.65	0.60	0.70	0.65	0.75
C	0.40	0.80	0.45	0.85	0.50	0.90

All orthopair constraints hold rungwise, e.g.,

$$\begin{aligned}
\mathrm{A}: &\ 0.75^2+0.45^2=0.765,\ 0.80^3+0.40^3=0.576,\ 0.85^5+0.35^5\approx 0.449;\\
\mathrm{B}: &\ 0.55^2+0.65^2=0.725,\ 0.60^3+0.70^3=0.559,\ 0.65^5+0.75^5\approx 0.353;\\
\mathrm{C}: &\ 0.40^2+0.80^2=0.800,\ 0.45^3+0.85^3\approx 0.705,\ 0.50^5+0.90^5\approx 0.622.
\end{aligned}$$

Using the rung-averaged score $S(x)=\frac{1}{3}\sum_{q\in\{2,3,5\}}(\mu_q(x)^q-\nu_q(x)^q)$, we obtain $S(\mathrm{A})>S(\mathrm{B})>S(\mathrm{C})$, so A is creditworthy across all postures; B becomes borderline (acceptable only for more permissive q); and C remains below threshold even when policies are lenient. The MROFS representation thus supports *policy-aware* decisions without recomputing models for each posture.Specialization.

Theorem 2 (MultiRung Orthopair Fuzzy Sets generalize q-ROFS)**.** *Let X be a nonempty universe and let $q\ge 1$.*

(A) *The class of MultiRung Orthopair Fuzzy Sets (MROFS) on X with the singleton rung set $R=\{q\}$ is in bijection with the class of q-rung orthopair fuzzy sets (q-ROFS) on X. Explicitly, an MROFS*

$$\mathcal{A}=\{\langle x,(\mu_q(x),\nu_q(x))\rangle:\ x\in X\},\qquad (\mu_q(x))^q+(\nu_q(x))^q\le 1,$$

is exactly a q-ROFS, and conversely every q-ROFS arises in this way.

(B)Embedding into any rung family containing q. *Let R be any finite set of rungs with $q\in R$. Define*

$$\iota_{q\to R}:\ \{q\text{-}ROFS\ on\ X\}\ \longrightarrow\ \{MROFS\ on\ (X,R)\}$$

as follows: for a q-ROFS given by $(\mu,\nu):X\to[0,1]^2$ with $\mu(x)^q+\nu(x)^q\le 1$, set, for each $x\in X$ and $r\in R$,

$$\mu_r(x):=\mu(x)^{q/r},\qquad \nu_r(x):=\nu(x)^{q/r}.$$

Then $\iota_{q\to R}(\mu,\nu)$ is an MROFS on (X,R) and the map $\iota_{q\to R}$ is injective.

(C)Rungwise constraint is preserved. *For every $x\in X$ and every $r\in R$ constructed in* (B)*,*

$$\big(\mu_r(x)\big)^r+\big(\nu_r(x)\big)^r=\mu(x)^q+\nu(x)^q\ \le\ 1,$$

so the orthopair admissibility holds simultaneously at all rungs $r\in R$.

Proof: *(A)* With $R = \{q\}$, the definition of an MROFS reads: assign to each $x \in X$ a pair $(\mu_q(x), \nu_q(x)) \in [0,1]^2$ with $(\mu_q(x))^q + (\nu_q(x))^q \leq 1$. This is precisely the definition of a q-ROFS. The converse identification is immediate, hence a bijection.

(B)–(C) Fix any q-ROFS (μ, ν) on X. For $r \in R$ define μ_r, ν_r by $\mu_r := \mu^{\,q/r}$ and $\nu_r := \nu^{\,q/r}$ (pointwise). Because $0 \leq \mu(x), \nu(x) \leq 1$, the power maps $t \mapsto t^{q/r}$ take $[0,1]$ into itself, so $\mu_r, \nu_r : [0,1]$. Moreover, for each $x \in X$,

$$\big(\mu_r(x)\big)^r + \big(\nu_r(x)\big)^r = \big(\mu(x)^{q/r}\big)^r + \big(\nu(x)^{q/r}\big)^r = \mu(x)^q + \nu(x)^q \leq 1,$$

verifying the rungwise orthopair constraint and establishing that $\iota_{q\to R}(\mu, \nu)$ is an MROFS on (X, R).

Injectivity of $\iota_{q\to R}$ follows by reading off the $r = q$ coordinate: for all x, we have $\mu_q(x) = \mu(x)$ and $\nu_q(x) = \nu(x)$. Hence if two q-ROFSs map to the same MROFS, their q-coordinates coincide, and so they are identical. This proves both (B) and (C).

3.1 | MultiLinguistic Fuzzy Set

A MultiLinguistic Fuzzy Set represents each element simultaneously across multiple linguistic domains, integrating diverse vocabularies and hedges for richer, multi-granular semantic reasoning.

Definition 10 (MultiLinguistic Fuzzy Set (MLFS))**.** Let J be a finite index set of *linguistic layers* (or *granularities*). For each $j \in J$, let

$$\mathbb{L}_j = (S_j, \preceq_j, H_j), \qquad \Sigma_j : S_{j,H_j} \to \mathsf{F}(X)$$

be a linguistic domain and its semantics on X (with S_{j,H_j} the hedge-closure of S_j under H_j). A *MultiLinguistic Fuzzy Set* on X is the collection

$$\widetilde{\mathcal{A}} \;=\; \Big\{ \big\langle x,\ \{\, \mu_\ell^{(j)}(x)\, \}_{\ell \in S_{j,H_j},\ j \in J} \big\rangle \;:\; x \in X \Big\},$$

where, for each $j \in J$ and $\ell \in S_{j,H_j}$, $\mu_\ell^{(j)}(x) = \Sigma_j(\ell)(x) \in [0,1]$. Thus every $x \in X$ is described *simultaneously* in multiple linguistic structures $\{\mathbb{L}_j\}_{j \in J}$ (e.g., coarse vs. fine vocabularies, domain-specific lexicons).

Example 11 (MLFS in smart agriculture: field moisture assessment)**.** Let X be farmland plots and consider two linguistic layers on X. Layer $j = 1$ (farmer lexicon): $S_1 = \{\text{dry}, \text{workable}, \text{wet}\}$. Layer $j = 2$ (sensor lexicon): $S_2 = \{\text{low}, \text{medium}, \text{high}\}$ moisture. Semantics Σ_1, Σ_2 map each label to a fuzzy set over the actual volumetric water content $m \in [0,1]$ (e.g., triangular/trapezoidal shapes).

For plot x with measured $(m, \text{soil temp}) = (0.42,\ 18°\text{C})$, suppose

$$\mu^{(1)}_{\text{dry}}(x) = 0.05, \quad \mu^{(1)}_{\text{workable}}(x) = 0.80, \quad \mu^{(1)}_{\text{wet}}(x) = 0.20,$$

$$\mu^{(2)}_{\text{low}}(x) = 0.10, \quad \mu^{(2)}_{\text{medium}}(x) = 0.75, \quad \mu^{(2)}_{\text{high}}(x) = 0.15.$$

The MultiLinguistic Fuzzy Set (MLFS) stores both layers simultaneously: $\{\mu_\ell^{(1)}(x)\}_{\ell \in S_1}$ and $\{\mu_\ell^{(2)}(x)\}_{\ell \in S_2}$, supporting agronomic and sensor-centric reasoning within the same representation.

Example 12 (MLFS in urban traffic monitoring: perceived vs. operational congestion)**.** Let X be road segments. Layer $j = 1$ (driver perception): $S_1 = \{\text{light}, \text{moderate}, \text{heavy}\}$ congestion. Layer $j = 2$ (operations lexicon): $S_2 = \{\text{LOS-A}, \text{LOS-C}, \text{LOS-E}\}$ (three representative service levels). Semantics Σ_1, Σ_2 are calibrated from flow (veh/h), occupancy (%), and average speed.

For segment x at peak hour (flow$=1850$ veh/h, speed$=28$ km/h), assume

$$\mu^{(1)}_{\text{light}}(x) = 0.05, \quad \mu^{(1)}_{\text{moderate}}(x) = 0.40, \quad \mu^{(1)}_{\text{heavy}}(x) = 0.70,$$

$$\mu^{(2)}_{\text{LOS-A}}(x) = 0.00, \quad \mu^{(2)}_{\text{LOS-C}}(x) = 0.35, \quad \mu^{(2)}_{\text{LOS-E}}(x) = 0.80.$$

The MLFS captures both the human-facing and control-room views of the same physical state x, enabling downstream decisions to consider both layers.

Definition 11 (Cross-layer translation and aggregation)**.** A *translation* from layer j to k is a map $\mathcal{T}_{j\to k} : S_{j,H_j} \to \mathsf{F}(S_{k,H_k})$ (label-to-set on the target label space), whose semantics-induced pullback on X is

$$\forall\, \ell \in S_{j,H_j} : \quad \Sigma_{j\to k}(\ell)(x) \;:=\; \sup_{\lambda \in S_{k,H_k}} \Big(\mathcal{T}_{j\to k}(\ell)(\lambda) \;\wedge\; \Sigma_k(\lambda)(x) \Big),$$

with $\wedge$ a t-norm (e.g., minimum). A *global score* for x may be formed by a weighted aggregation over layers:

$$\mu^{\mathrm{glob}}(x) \;:=\; \mathrm{Agg}\Big(w_j \cdot \mathrm{Agg}_j\big(\{\mu^{(j)}_\ell(x)\}_{\ell\in S_{j,H_j}}\big) \;:\; j \in J \Big), \qquad w_j \geq 0,\ \sum_j w_j = 1,$$

where Agg_j and Agg are monotone averaging operators.

Example 13 (MLFS in smart agriculture: field moisture assessment)**.** Let X be farmland plots and consider two linguistic layers on X. Layer $j = 1$ (farmer lexicon): $S_1 = \{\mathrm{dry}, \mathrm{workable}, \mathrm{wet}\}$. Layer $j = 2$ (sensor lexicon): $S_2 = \{\mathrm{low}, \mathrm{medium}, \mathrm{high}\}$ moisture. Semantics Σ_1, Σ_2 map each label to a fuzzy set over the actual volumetric water content $m \in [0, 1]$ (e.g., triangular/trapezoidal shapes).

For plot x with measured $(m, \mathrm{soil\ temp}) = (0.42,\ 18^\circ\mathrm{C})$, suppose

$$\mu^{(1)}_{\mathrm{dry}}(x) = 0.05, \quad \mu^{(1)}_{\mathrm{workable}}(x) = 0.80, \quad \mu^{(1)}_{\mathrm{wet}}(x) = 0.20,$$
$$\mu^{(2)}_{\mathrm{low}}(x) = 0.10, \quad \mu^{(2)}_{\mathrm{medium}}(x) = 0.75, \quad \mu^{(2)}_{\mathrm{high}}(x) = 0.15.$$

The MultiLinguistic Fuzzy Set (MLFS) stores both layers simultaneously: $\{\mu^{(1)}_\ell(x)\}_{\ell\in S_1}$ and $\{\mu^{(2)}_\ell(x)\}_{\ell\in S_2}$, supporting agronomic and sensor-centric reasoning within the same representation.

Example 14 (MLFS in urban traffic monitoring: perceived vs. operational congestion)**.** Let X be road segments. Layer $j = 1$ (driver perception): $S_1 = \{\mathrm{light}, \mathrm{moderate}, \mathrm{heavy}\}$ congestion. Layer $j = 2$ (operations lexicon): $S_2 = \{\mathrm{LOS\text{-}A}, \mathrm{LOS\text{-}C}, \mathrm{LOS\text{-}E}\}$ (three representative service levels). Semantics Σ_1, Σ_2 are calibrated from flow (veh/h), occupancy (%), and average speed.

For segment x at peak hour (flow= 1850 veh/h, speed= 28 km/h), assume

$$\mu^{(1)}_{\mathrm{light}}(x) = 0.05, \quad \mu^{(1)}_{\mathrm{moderate}}(x) = 0.40, \quad \mu^{(1)}_{\mathrm{heavy}}(x) = 0.70,$$
$$\mu^{(2)}_{\mathrm{LOS\text{-}A}}(x) = 0.00, \quad \mu^{(2)}_{\mathrm{LOS\text{-}C}}(x) = 0.35, \quad \mu^{(2)}_{\mathrm{LOS\text{-}E}}(x) = 0.80.$$

The MLFS captures both the human-facing and control-room views of the same physical state x, enabling downstream decisions to consider both layers.

Example 15 (Cross-layer translation & aggregation: temperature comfort (fine $\to$ coarse))**.** Let $X = \mathbb{R}$ (indoor temperature). Layer $j = 1$ (coarse): $S_1 = \{\mathrm{cold}, \mathrm{warm}, \mathrm{hot}\}$, semantics Σ_1. Layer $j = 2$ (fine): $S_2 = \{\mathrm{cool}, \mathrm{mild}, \mathrm{warm}\}$, semantics Σ_2. Define a translation $\mathcal{T}_{2\to 1} : S_2 \to \mathsf{F}(S_1)$ by the (label-to-label) matrix

$\mathcal{T}_{2\to 1}$	cold	warm	hot
cool	0.9	0.2	0.0
mild	0.2	0.8	0.0
warm	0.0	0.6	0.7

(interpreted as fuzzy memberships over S_1). For $x = 23^\circ\mathrm{C}$, suppose

$$\mu^{(2)}_{\mathrm{cool}}(x) = 0.10, \quad \mu^{(2)}_{\mathrm{mild}}(x) = 0.75, \quad \mu^{(2)}_{\mathrm{warm}}(x) = 0.30.$$

The pullback of the fine label "mild" to the coarse layer is

$$\Sigma_{2\to 1}(\mathrm{mild})(x) = \sup_{\lambda\in S_1} \big(\mathcal{T}_{2\to 1}(\mathrm{mild})(\lambda) \wedge \Sigma_1(\lambda)(x)\big),$$

and similarly for "cool" and "warm" (with $\wedge = \min$). To form a global score, let $w_1 = 0.4$, $w_2 = 0.6$ and choose $\mathrm{Agg}_1 = \mathrm{height}(\cdot)$ on $\{\mu^{(1)}_\ell(x)\}_{\ell\in S_1}$, $\mathrm{Agg}_2 = \max\{\mu^{(2)}_{\mathrm{cool}}(x), \mu^{(2)}_{\mathrm{mild}}(x), \mu^{(2)}_{\mathrm{warm}}(x)\} = 0.75$, then

$$\mu^{\mathrm{glob}}(x) = 0.4 \cdot \mathrm{Agg}_1(\cdot) + 0.6 \cdot 0.75,$$

yielding a single comfort score that consistently blends coarse and fine lexicons via translation and weighted aggregation.

Example 16 (Cross-layer translation & aggregation: sentiment (social → business scale))**.** Let X be product mentions. Layer $j = 1$ (business triad): $S_1 = \{\text{negative}, \text{neutral}, \text{positive}\}$. Layer $j = 2$ (social-media lexicon): $S_2 = \{\text{awful}, \text{bad}, \text{okay}, \text{good}, \text{awesome}\}$. A translation $\mathcal{T}_{2\to1}$ is specified by

$\mathcal{T}_{2\to1}$	negative	neutral	positive
awful	0.95	0.05	0.00
bad	0.80	0.20	0.00
okay	0.10	0.80	0.10
good	0.00	0.20	0.80
awesome	0.00	0.05	0.95

For mention x, suppose the social classifier outputs

$$\mu^{(2)}_{\mathrm{a}}\quad(x) = 0.00,\ \mu^{(2)}_{\mathrm{bad}}(x) = 0.10,\ \mu^{(2)}_{\mathrm{okay}}(x) = 0.30,\ \mu^{(2)}_{\mathrm{good}}(x) = 0.65,\ \mu^{(2)}_{\mathrm{awesome}}(x) = 0.20.$$

The pullback to the business layer for, e.g., "positive" is

$$\Sigma_{2\to1}(\cdot)(x):\ \mu^{(1)}_{\mathrm{positive}}(x) = \sup_{\lambda\in S_2}\big(\mathcal{T}_{2\to1}(\lambda)(\text{positive})\wedge\mu^{(2)}_{\lambda}(x)\big) = \max\{0, 0, 0.10\wedge0.30, 0.80\wedge0.65, 0.95\wedge0.20\} = 0.65.$$

Similarly,

$$\mu^{(1)}_{\mathrm{neutral}}(x) = \max\{0.05\wedge0, 0.20\wedge0.10, 0.80\wedge0.30, 0.20\wedge0.65, 0.05\wedge0.20\} = 0.30,$$

$$\mu^{(1)}_{\mathrm{negative}}(x) = \max\{0.95\wedge0, 0.80\wedge0.10, 0.10\wedge0.30, 0\wedge0.65, 0\wedge0.20\} = 0.10.$$

With layer-weights $w_1 = 0.7,\ w_2 = 0.3$ and Agg_1 the signed score $S_1 = \mu^{(1)}_{\mathrm{positive}}(x) - \mu^{(1)}_{\mathrm{negative}}(x) = 0.55$, Agg_2 the social polarity $S_2 = (\mu^{(2)}_{\mathrm{good}} + \mu^{(2)}_{\mathrm{awesome}}) - (\mu^{(2)}_{\mathrm{bad}} + \mu^{(2)}_{\mathrm{awful}}) = 0.75$, the global sentiment score is

$$\mu^{\mathrm{glob}}(x) = 0.7\cdot0.55 + 0.3\cdot0.75 = 0.61,$$

which translates fine-grained social signals to the business triad and fuses both views.

4 | Conclusion

In this paper, we studied the MultiType, MultiRung Orthopair, and MultiLinguistic Fuzzy Sets, which extend the type, q-rung orthopair, and linguistic frameworks into multi-structured forms. In the future, we expect further research to focus on designing algorithms for these concepts and on developing extensions that employ Neutrosophic Sets[20, 21, 22], HyperFuzzy Sets[23], Quadripartitioned Neutrosophic Sets[24, 25], and Plithogenic Sets [26, 27, 28].

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Data Availability

This manuscript presents purely conceptual work without empirical data. Scholars interested in these ideas are invited to undertake experimental or case-study research to substantiate and extend the proposed frameworks.

Ethical Approval

This paper involves no human or animal subjects and thus did not require ethics committee review or approval.

Use of Generative AI and AI-Assisted Tools

We use generative AI and AI-assisted tools for tasks such as English grammar checking, and We do not employ them in any way that violates ethical standards.

Conflicts of Interest

The authors declare that there are no competing interests concerning the content or publication of this article.

Disclaimer

The theoretical models and propositions herein have not yet been subjected to practical validation. Readers should independently verify all citations and be aware that inadvertent inaccuracies may remain. The opinions expressed are those of the authors and do not necessarily represent the views of affiliated organizations.

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