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Development of a Modified Fuzzy Data Envelopment Analysis Model Based on Uncertainty Modeling and a Novel Efficiency Function

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Abstract

Data Envelopment Analysis (DEA) is a well-established nonparametric method for evaluating the relative efficiency of Decision-Making Units (DMUs). Conventional DEA models assume deterministic input and output data, which is rarely the case in real-world applications characterized by uncertainty and ambiguity. To overcome this limitation, this paper proposes a new fuzzy DEA framework by developing a modified efficiency function that is able to explicitly incorporate data uncertainty through fuzzy numbers. A new weighted α cut-based efficiency measure is introduced, which transforms the fuzzy DEA model into an equivalent deterministic linear programming formulation. The theoretical properties of the proposed model, including feasibility, uniformity, and boundedness, are investigated. Numerical experiments based on simulated data demonstrate the effectiveness and robustness of the proposed approach compared with classical DEA and existing fuzzy DEA models. The proposed framework provides a flexible and reliable tool for evaluating efficiency in uncertain environments.

Keywords: Data envelopment analysis, Fuzzy data envelopment analysis, Uncertainty modeling, Efficiency measurement, Linearization.

1 | Introduction

Efficiency assessment plays a vital role in performance analysis and decision-making in various sectors such as banking, healthcare, manufacturing, energy, and education [1–4]. Data Envelopment Analysis (DEA), originally introduced by Charnes et al. [5], has become one of the most widely used techniques for measuring

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the relative efficiency of homogeneous Decision-Making Units (DMUs) with multiple inputs and outputs. Its theoretical and methodological advances have been widely documented in the literature over the past decades [6], [7]. Despite its widespread application, conventional DEA models rely on the assumption that all input and output data are deterministic and precisely measured. However, in many real-world situations, data are subject to uncertainty due to measurement errors, incomplete information, environmental variability, and random fluctuations [8], [9]. Ignoring such uncertainty may lead to biased efficiency scores and unreliable benchmark results. To overcome this shortcoming, fuzzy set theory has been widely integrated with DEA to model ambiguous and uncertain data. Comprehensive reviews of fuzzy DEA models, including probability and likelihood-based approaches, are presented in [10]. Since the pioneering developments of fuzzy DEA, various models based on α-cuts, probabilistic programming, and validity criteria have been proposed [11], [12]. Developments have incorporated undesirable outcomes and ideal point concepts into fuzzy environments [13], fuzzy stochastic data structures [14], [15], and robust fuzzy perturbation frameworks [16], [17]. Fully Fuzzy DEA formulations and slack-based measurement structures have also been developed to increase modeling flexibility [18], while bi-objective fuzzy DEA models have been introduced to address multiple performance objectives simultaneously [13].

Furthermore, fuzzy and interval DEA approaches have been extended to network structures, inverse DEA problems, and ordinal or interval data settings [9], [19–21]. Applications under data ambiguity have been reported in banking and financial systems [3], environmental and energy efficiency assessment [2], [22], [23], [24], healthcare systems [1], [16], and collaborative or game-based manufacturing structures [25], [26]. However, many existing fuzzy DEA approaches suffer from high computational complexity, especially when multiple α-level optimization problems must be solved or when probabilistic and mixed-uncertainty structures are included [8], [10], [12].

In addition, some models face challenges in interpretability and integrating efficiency under uncertainty. Motivated by these challenges, this paper proposes a new fuzzy DEA model by developing a modified efficiency function that considers uncertainty in a systematic and computationally efficient manner. The main contributions of this study are summarized as follows:

I. A new fuzzy efficiency function based on weighted α-cuts is introduced.

II. The classical Charnes–Cooper–Rhodes (CCR) model is generalized to a fuzzy environment with a linearizable formulation.

III. The theoretical properties of the proposed model are analyzed.

IV. Numerical experiments using simulated data validate the proposed approach.

The rest of the paper is organized as follows. Section 2 reviews the relevant literature. Section 3 introduces the basic concepts of DEA and fuzzy numbers. Section 4 presents the proposed fuzzy DEA model. Section 5 reports the numerical results. Finally, Section 6 concludes the paper.

2|Literature Review

Since its introduction by Charnes et al. [5], DEA has become a mature and widely used performance evaluation methodology. Over the past five decades, extensive theoretical developments and practical applications have been documented, reflecting the methodological richness and adaptability of DEA to various decision-making environments [6], [7]. A comprehensive review of the historical evolution and future directions of DEA is provided in [6], which emphasizes methodological developments such as network DEA, dynamic DEA, hyperefficient models, and robustness analysis. Recent advances have focused on incorporating complex production structures, undesirable outputs, and multi-stage systems. Network-based models and SBM have been proposed for evaluating serial and parallel processes as well as environmental performance [27–29]. Cooperative game-based and cost allocation approaches have further expanded the applications of DEA in production and environmental management [25], [26], [30], [31]. Hyper-efficient and environmentally oriented models have also been developed to address considerations of limited availability and NetZero [24]. In addition, DEA models with strong joint weight validation have been introduced to increase the discriminant power and ensure meaningful weight structures [32]. Applications have been extended to energy planning, productivity analysis, and socio-economic efficiency assessment [4], [23], [33]. Despite these advances, classical DEA assumes that all inputs and outputs are deterministic and precisely observed. However, in many real-world applications, such as healthcare, banking, agriculture, and energy systems, data are inherently uncertain due to measurement errors, environmental variability, and incomplete information. To address this limitation, uncertainty modeling in DEA has attracted considerable attention. A major research stream integrates fuzzy set theory into DEA. Early fuzzy DEA models relied on α-cut representations, likelihood and necessity criteria, and robust probabilistic programming frameworks to transform fuzzy programs into deterministic equivalents [12]. A structured and comprehensive review of fuzzy DEA models, including probability-based and probability-based approaches, is presented in [10]. Adjustable fuzzy DEA models were proposed to accommodate optimism and pessimism in efficiency evaluation [11], while fuzzy clustering and collaborative frameworks have further enriched this methodology [34]. More recent contributions incorporate advanced uncertainty structures.

For example, fuzzy random variables and mixed uncertainty frameworks have been introduced to simultaneously model randomness and fuzziness [14], [15]. Robust fuzzy DEA models have improved robustness to data fluctuations by using degrees of disturbance and uncertainty boxes [16], [17]. Adjustable-probability fuzzy network DEA models have further increased flexibility in financial performance and investment evaluation [19], while cost efficiency in stochastic environments has been investigated through finite-probability DEA formulations [8]. The applications of fuzzy DEA have expanded significantly in various domains. In healthcare performance evaluation, fuzzy DEA has been used to construct more reliable indicators of quality and access [1]. In banking and finance, two-stage fuzzy approaches have addressed data ambiguity and uncertainty [3]. Environmental and energy efficiency analyses have considered undesirable outputs and fuzzy modeling to better capture real-world production conditions [2], [22], [35].

In addition, recent developments include bi-objective fuzzy DEA models for multi-criteria efficiency evaluation [13], fully fuzzy DEA with incremental slack-based criteria [18], and inverse or interval-based DEA approaches under fuzzy and integer uncertainty [9], [20], [21]. These models increase flexibility and differentiability, but often increase computational complexity or require complex probabilistic interpretations. Although significant progress has been made, several challenges remain. Many fuzzy DEA models require solving multiple optimization problems at different α levels, which can result in a high computational burden. Others lack an integrated efficiency function that systematically captures uncertainty information while maintaining linear programmability and interpretability. Therefore, there is a need for a computationally efficient and theoretically sound fuzzy DEA framework that directly embeds uncertainty into the efficiency function. Addressing these research gaps, the present study proposes a modified fuzzy DEA model based on a novel weighted alpha-cut efficiency function. The proposed framework provides a linearizable formulation, preserves the structure of the classical CCR model, and ensures desirable theoretical properties, including feasibility, boundedness, and uniformity.

3|Preliminary

To develop the proposed fuzzy DEA framework, it is necessary to briefly review the basic concepts of classical DEA and its underlying mathematical formulation. DEA is a nonparametric, frontier-based optimization technique used to assess the relative efficiency of homogeneous DMUs that consume multiple inputs to produce multiple outputs. Since its introduction by Charnes et al. [5], the CCR model has become the cornerstone of efficiency measurement assuming constant returns to scale. The classical DEA framework constructs a piecewise linear production frontier from the observed data and measures the relative efficiency of each DMU relative to this frontier. An efficiency score of 1 indicates that the DMU is on the efficient frontier. In contrast, a score less than one indicates inefficiency and a proportional reduction in inputs (in the input-driven case) without a reduction in output level. In the following subsection, the input-driven CCR model, which forms the basis of the proposed fuzzy expansion, is presented.

3.1 | The Classical Data Envelopment Analysis Model

Consider n DMUs, each of which uses m inputs to produce s outputs. Let x_{ij} denote the amount of input I used by DMU_j and y_{rj} denote the amount of output r produced by DMU_j. The input-driven CCR model is given by:

$$\min_{\theta,\lambda} \theta. \tag{1}$$

s.t.

$$\sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{i,o}, \quad i = 1, \dots, m. \tag{2}$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro}, \quad r = 1, \dots, s. \tag{3}$$

$$\lambda_j \geq 0, \quad j = 1, \dots, n. \tag{4}$$

The optimal value $\theta \in (0,1]$ represents the relative efficiency of DMU_o.

3.2 | Fuzzy Numbers and α Cuts

In many real-world applications, input and output data are not precise due to uncertainty, ambiguity, or incomplete information. To model such situations, fuzzy set theory is incorporated into DEA [10–13], [18], [20], [35]. A triangular fuzzy number $\tilde{x}$ is represented as $\tilde{x} = (x^l, x^m, x^u)$, where x^l, x^m and x^u denote the low, mode, and high values, respectively. The α cut of $\tilde{x}$ is defined as:

$$\tilde{x}^{(\alpha)} = [x^l + \alpha\,(x^m - x^l),\ x^u - \alpha\,(x^u - x^m)],\ \alpha \in [0, 1]. \tag{5}$$

The α-cuts representation enables transforming fuzzy optimization problems into interval or deterministic equivalents [8], [9], [13–16], [18], [21].

4 | Proposed Modified Fuzzy Data Envelopment Analysis Model

This section introduces a new fuzzy DEA model by proposing a modified efficiency function that explicitly accounts for data uncertainty via fuzzy numbers and confidence-level weighting. Unlike conventional fuzzy DEA models that rely on fixed α cutoff levels [10–13], the proposed framework integrates information across the entire confidence spectrum, leading to a more robust and flexible efficiency assessment.

Consider a set of n DMUs, where each DMU_j $(j = 1, \dots, n)$ consumes m fuzzy inputs $\tilde{x}_{ij} (i = 1, \dots, m)$ to produce s fuzzy outputs $\tilde{y}_{rj}$. It is assumed that all fuzzy inputs and outputs are represented by convex and normal fuzzy numbers. Using the α-cuts representation, the fuzzy input and output values at the confidence level $\alpha \in [0, 1]$ are represented by ${x_{ij}}^{(\alpha)}$ and ${y_{rj}}^{(\alpha)}$, respectively.

4.1 | Modified Fuzzy Efficiency Function

The fuzzy efficiency of the DMU under evaluation, denoted by DMU, is defined as a weighted supremum over all cutoff levels α:

$$\widetilde{E}_o = \sup_{\alpha \in [0,1]} \left\{ \frac{\sum_{r=1}^{s} u_r y_{r_o}^{(\alpha)}}{\sum_{i=1}^{m} v_i x_{io}^{(\alpha)}} \cdot \omega(\alpha) \right\}, \tag{6}$$

where u_r and v_i denote the output and input weights, respectively, and $\omega(\alpha)$ is a confidence weighting function introduced to reflect the relative importance of different levels of uncertainty.

The weighting function $\omega(\alpha)$ satisfies the following conditions:

$$\omega(\alpha) \geq 0, \quad \int_0^1 \omega(\alpha)\, d\alpha = 1. \tag{7}$$

This formulation allows decision makers to emphasize higher confidence levels (e.g., larger α values) or to account for risk attitudes by appropriately choosing the functional form of $\omega(\alpha)$. The normalization condition in *Eq. (7)* ensures that the proposed efficiency measure remains constant over scale and comparable across DMUs. Furthermore, by appropriately specifying the functional form of $\omega(\alpha)$, the proposed model can accommodate different risk attitudes of decision makers, ranging from conservative (with an emphasis on higher levels of α) to neutral risk preferences or risk seeking [8], [10], [11].

4.2 | Discussion of the Proposed Formulation

The proposed efficiency measure in *Eq. (1)* offers several important advantages over existing fuzzy DEA models: 1) it avoids the reliance on one or a limited number of cutoff levels α and thus captures the full uncertainty structure of fuzzy data, 2) the inclusion of $\omega(\alpha)$ provides a flexible mechanism for modeling decision makers' preferences towards uncertainty, and 3) the proposed formulation preserves the core philosophy of classical DEA [10–13], [18] while extending its applicability to uncertain environments [6], [7].

4.3 | Discretization and Linearization Procedure

To obtain a computationally tractable model, the interval $\alpha \in [0, 1]$ is discretized into K confidence levels $\{\alpha_1, \alpha_2, \ldots, \alpha_K\}$ with corresponding weights $\omega_m = \omega(\alpha_m)$, satisfying $\sum_{m=1}^{K} \omega_m = 1$. Accordingly, the fuzzy efficiency evaluation problem in *Eq. (6)* can be approximated as follows:

$$\max \sum_{m=1}^{K} \omega_m \frac{\sum_{r=1}^{s} u_r y_{r_o}^{(\alpha_m)}}{\sum_{i=1}^{m} v_i x_{io}^{(\alpha_m)}}. \tag{8}$$

By applying the Charnes et al. [5] transformation, the above fractional programming problem can be converted into an equivalent linear programming model, and its computational efficiency and solvability are ensured using standard DEA software packages [6], [7].

In the next section, the theoretical properties of the proposed model, including feasibility, boundedness, and uniformity, are formally analyzed.

4.4 | Theoretical Propositions and Proofs

Proposition 1 (feasibility). For each confidence level $\alpha \in [0, 1]$ and each DMU_o, the proposed α-level CCR coverage model is feasible. Consider the input-oriented CCR coverage constraints at the α level:

$$\sum_{j=1}^{n} \lambda_j x_{ij}^{c}(\alpha) \leq \theta x_{io}^{c}(\alpha), \quad \sum_{j=1}^{n} \lambda_j y_{rj}^{c}(\alpha) \geq y_{ro}^{c}(\alpha), \quad \lambda_j \geq 0. \tag{9}$$

Choose $\lambda_o = 1$ and $\lambda_j = 1$ for $j \neq o$, and set $\theta = 1$. Then for all inputs i **(10)**

$$\sum_{j} \lambda_j x_{ij}^{c}(\alpha) = \theta x_{io}^{c}(\alpha) \leq 1.\, x_{io}^{c}(\alpha). \tag{11}$$

And for all outputs r:

$$\sum_{j} \lambda_j y_{rj}^{c}(\alpha) = y_{ro}^{c}(\alpha) \geq y_{io}^{c}(\alpha). \tag{12}$$

Thus, a feasible solution exists for every DMU and every α [6], [7].

Proposition 2 (boundedness and range). For any $\alpha \in [0,1]$, the optimal efficiency score $\theta_o(\alpha)$ of the input-driven CCR covering model satisfies $0 < \theta_o(\alpha) \leq 1$.

The feasibility is guaranteed by Proposition 1, hence there exists an optimal solution (or an infimum is obtained under standard Linear Programming (LP) assumptions). Since the objective is min θ and $\theta = 1$ is feasible, we have $\theta_o(\alpha) \leq 1$.

To show positivity, note that all inputs are assumed to be nonnegative, and at least one input of DMU_o is strictly positive (a standard DEA requirement). If $\theta = 0$ is possible, then the constraints imply that $P_j\, \lambda_j x^c_{ij}(\alpha) \leq 0$ for all i, which requires $\lambda_j = 0$ for all j if there is a strictly positive input to each DMU, but then the output constraints fail since $y^c_{ro}(\alpha) > 0$ for at least one output. Hence $\theta_o(\alpha) > 0$.

In the input-driven CCR model, the efficiency value of each decision-making unit is interpreted as follows:

In the input-driven CCR model, the optimal efficiency value θ is in the interval $\theta \in (0,1]$. The closer the value of θ is to 1, the more efficient the decision-making unit is. The value $\theta = 1$ indicates complete efficiency, and the unit is on the efficient frontier. In contrast, values less than 1 indicate relative inefficiency and the potential for a proportional reduction in inputs without a reduction in output levels [6], [7].

Proposition 3 (monotonicity). Suppose DMU_o is changed to DMU'_o such that $x^c_{io'}(\alpha) \leq x^c_{io}(\alpha)$ for all i, $y^c_{ro'}(\alpha) \geq y^c_{ro}(\alpha)$ for all r. Then the optimal efficiency of input-oriented CCR [6], [32]. satisfies $\theta_{o'}(\alpha) \geq \theta_o(\alpha)$. Let (λ^*, θ^*) be an optimal solution for DMU_o at level α. Consider the same λ^* for DMU'_o. The input constraints become:

$$\sum_j \lambda^* x^c_{ij}(\alpha)\, X \leq \theta^*\, x^c_{io}(\alpha) \leq \theta^* x^c_{io'}(\alpha) . \frac{x^c_{io}(\alpha)}{x^c_{io'}(\alpha)}. \tag{13}$$

And since $x^c_{io'}(\alpha) \leq x^c_{io}(\alpha)$ the RHS for DMU'_o is not smaller for the same θ^*. The output constraints become easier because $y^c_{ro'}(\alpha) \geq y^c_{ro}(\alpha)$; hence (λ^*, θ^*) is feasible for DMU'_oor can be made feasible with $\theta \geq \theta^*$. So, the optimal value for DMU'_o cannot be smaller than θ^*.

$$\theta_{o'}(\alpha) \geq \theta_o(\alpha).$$

Proposition 4 (consistency with Crisp CCR). If the uncertainty becomes zero, the model reverts exactly to the classical CCR. If all the fuzzy inputs and outputs are transformed into deterministic values, i.e., $x^L_{rj}(\alpha) = x^U_{rj}(\alpha) = x_{ij}$ and $y^L_{rj}(\alpha) = y^U_{rj}(\alpha) = y_{rj}$ for all α, then $\theta_o(\alpha)$ is independent of α and equal to the efficiency of the classical CCR DMU_o. Under this transformation, $x^C_{ij}(\alpha) = x_{ij}$ and $y^C_{rj}(\alpha) = y_{rj}$ for all α. Hence, the LP of the α level is the same for all α and equal to the LP of the standard CCR. Therefore, the optimal value equals the classical CCR score [6], [7].

Proposition 5 (aggregated efficiency well defined and bounded). Begin proposition let $\{\alpha_m\}^K_{m=1}$ be the discretization points with weights $\omega_m \geq 0$ and

$\sum^K_{m=1} \omega_m = 1$. Define the aggregate efficiency for DMUo as [7]:

$$E_o = \sum_{m=1}^{K} \omega_m\, \theta_o(\alpha_m). \tag{14}$$

Then $0 < E_o \leq 1$. From Proposition 2, for every k, we have $0 < \theta_o(\alpha_m) \leq 1$, Since $\omega_m \geq 0$.

and their sum is 1, E_o is a convex combination of numbers in $(0, 1]$. Therefore, $0 < E_o \leq 1$.

4.5 | Linear Programming Formulation of the Proposed α Level Fuzzy Charnes–Cooper–Rhodes Model

For each confidence level α_k, define the conservative realizations:

$$x_{ij}^{c}(\alpha_k) = x_{ij}^{U}(\alpha_k), \quad y_{rj}^{c}(\alpha_k) = y_{rj}^{L}(\alpha_k). \tag{15}$$

The proposed input-oriented fuzzy CCR covering model at the level α_k is as follows:

$$\min_{\theta_k,\lambda} \ \theta_k \tag{16}$$

s.t:

$$\sum_{j=1}^{n} \lambda_j x_{ij}^{C}(\alpha_k) \le \theta_k x_{io}^{C}(\alpha_k), \quad i = 1, \ldots, m. \tag{17}$$

$$\sum_{j=1}^{n} \lambda_j y_{rj}^{C}(\alpha_k) \ge y_{ro}^{C}(\alpha_k), \quad r = 1, \ldots, s. \tag{18}$$

$$\lambda_j \ge 0, \quad j = 1, \ldots, n. \tag{19}$$

The optimal value of $\theta_k(\alpha_k)$ is obtained by solving equations *Eq. (15)-(18).*

4.6 | Weighted Aggregation over the Confidence Spectrum

After solving the LP at each discrete confidence level α_k, the final efficiency score for the DMU_o is calculated as

$$E_o = \sum_{m=1}^{K} \omega_m \theta_o(\alpha_m). \tag{20}$$

where $\omega_m = \omega(\alpha_m)$ and $\sum_{m=1}^{K} \omega_m = 1$. This aggregation provides an interpretable efficiency score that integrates performance information across levels of uncertainty [10], [13], [18].

4.7 | Recommended Choices of Weighting Function

A risk-neutral decision maker may use a uniform weighting:

$$\omega(\alpha) = 1. \tag{21}$$

A conservative decision maker [8], [9], [11] may emphasize higher confidence levels via:

$$\omega(\alpha) = \frac{(p + 1)\alpha^p}{1}, \quad p > 0, \tag{22}$$

which satisfies:

$$\int_0^1 \omega(\alpha) d\alpha = 1. \tag{23}$$

Alternatively, an exponential emphasis can be defined as

$$\omega(\alpha) = \frac{\beta e^{\beta\alpha}}{e^{\beta} - 1}, \quad \beta > 0. \tag{24}$$

5 | Numerical Experiments and Management Analysis

This section evaluates the proposed fuzzy DEA model through extensive numerical experiments. The objectives are threefold:

I. to demonstrate the behavior of the proposed efficiency measure under uncertainty.

II. to assess its robustness and discriminant power.

III. to provide managerial insights through interpretable results.

To evaluate the robustness and discriminating power of the proposed fuzzy DEA model, we generate synthetic DMU_s with a known input-output structure. Let n denote the number of DMU_s with m inputs and s outputs.

First, the deterministic baseline data is generated as follows:

$$x_{ij} \sim \mathcal{U}(a_i, b_i),\ y_{rj} \sim \mathcal{U}(c_r, d_r), \tag{25}$$

where U denotes a uniform distribution. Next, uncertainty is introduced by constructing triangular fuzzy numbers for each input and output:

$$\tilde{x}_{ij} = \left(x_{ij}\left(1-\delta_{ij}^{x}\right), x_{ij}, x_{ij}\left(1+\delta_{ij}^{x}\right)\right). \tag{26}$$

$$\tilde{y}_{rj} = \left(y_{rj}\left(1-\delta_{ij}^{y}\right), y_{rj}, y_{rj}\left(1+\delta_{ij}^{y}\right)\right), \tag{27}$$

where δ_{ij}^{x} and δ_{rj}^{y} are the uncertainty ranges sampled from:

$$\delta_{ij}^{x} \sim \mathcal{U}(0.05, 0.25), \quad \delta_{rj}^{y} \sim \mathcal{U}(0.05, 0.25). \tag{28}$$

For a triangular fuzzy number $\tilde{z} = (z^l, z^m, z^u)$, the cut α is:

$$\tilde{z}(\alpha) = \left[z^l + \alpha\left(z^m - z^l\right),\ z^u - \alpha\left(z^u - z^m\right)\right], \qquad \alpha \in [0,1]. \tag{29}$$

5.1 | Solution Algorithm

Let $\{\alpha_m\}_{m=1}^{K}$ be the discretization points and $\{\omega_m\}_{m=1}^{K}$ be the weights. The computational procedure is as follows:

I. Generate deterministic data (x_{ij}, y_{rj}) for n decision-making units (DMU_s).

II. Construct triangular fuzzy numbers. $\tilde{x}_{ij}$ and $\tilde{y}_{rj}$.

III. For each α_k:

- *Compute the α cutoff limits. $x_{ij}^{U}(\alpha_k)$ and $y_{rj}^{L}(\alpha_k)$.*
- *For each DMU_o, solve LPs (14)-(17) and store $\theta_o(\alpha_m)$.*

IV. Compute the aggregate efficiency E_o for each DMU_o using *Eq. (18).*

The overall complexity requires solving n×K linear programs, each with n+1 variables and (m+s) constraints, which is computationally efficient for typical DEA sample sizes.

5.1 | Results

Table 1 reports the α-level efficiencies and the aggregate score E_o for a simulated sample with

$n = 15, m = 2, s = 2$ and $\alpha = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$ using uniform weights $\omega_k = \frac{1}{K}$.

Table 1. Simulated α level CCR efficiencies and aggregate score (conservative scenario).

DMU	$\theta(0)$	$\theta(0.2)$	$\theta(0.4)$	$\theta(0.6)$	$\theta(0.8)$	$\theta(1)$	E_o
DMU_1	0.2929	0.2964	0.3001	0.3040	0.3082	0.3126	0.3024
DMU_2	0.2746	0.2727	0.2705	0.2681	0.2655	0.2627	0.2690
DMU_3	0.3200	0.3139	0.3186	0.3181	0.3177	0.3175	0.3185
DMU_4	0.2731	0.2808	0.2882	0.2954	0.3023	0.3023	0.2915
DMU_5	0.5661	0.5617	0.5573	0.5528	0.5482	0.5436	0.5550

Several observations can be drawn from *Table 1*: 1) the efficiency scores differ across α levels, suggesting that the ranking of DMU_s may depend on the confidence levels when uncertainty is explicitly modeled. This effect is particularly pronounced for DMU_s with larger fuzzy domains (uncertainty domains), 2) the aggregate score E_o provides an interpretable summary measure that balances performance across levels of uncertainty. In risk-neutral settings (uniform ω_k), E_o can be interpreted as a conservative efficiency average across the confidence spectrum, and 3) sensitivity analysis with respect to $\omega(\alpha)$ allows the model to capture risk attitudes. Emphasizing higher α levels typically reduces the impact of severe uncertainty and yields more stable benchmarking, whereas giving more weight to lower α levels yields a more accurate (conservative) assessment of efficiency.

5.3 | Probability-Based Fuzzy Data Envelopment Analysis Model

In the probability-based approach, fuzzy constraints are managed through probability criteria. The input constraint can be properly formulated as follows:

$$\mathrm{Pos}\left(\sum_{j=1}^{n}\lambda_j\,\tilde{x}_{ij} \le \theta\tilde{x}_{i0}\right) \ge \eta, \qquad \eta \in (0,1], \tag{29}$$

where:

I. $\mathrm{Pos}(\cdot)$ denotes the probability measure.

II. $\tilde{x}_{ij}$ and $\tilde{x}_{i0}$ are the fuzzy inputs.

III. η is a predefined confidence level (probability).

Similarly, the output constraint can be written as

$$\mathrm{Pos}\left(\sum_{j=1}^{n}\lambda_j\,\tilde{y}_{ij} \ge \tilde{y}_{i0}\right) \ge \eta, \qquad \eta \in (0,1]. \tag{30}$$

Although this approach explicitly considers uncertainty, the resulting model is generally nonlinear and strongly dependent on the chosen confidence level η, which may complicate interpretation and comparability.

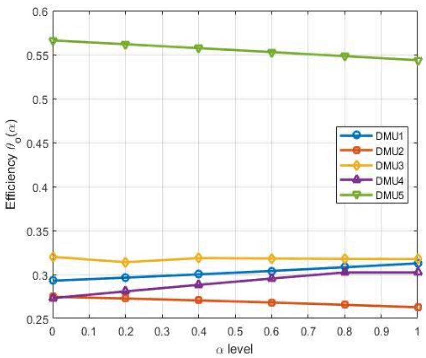

Fig. 1. Sensitivity analysis efficiency scores at α levels.

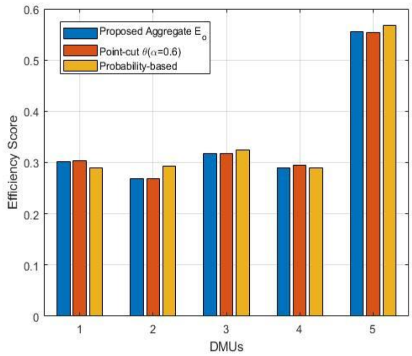

Fig. 2. Comparative evaluation of sum, cut-point, and probability-based DEA models.

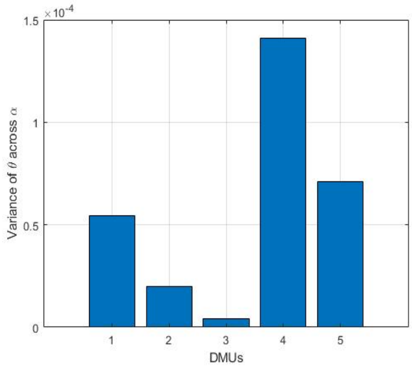

Fig. 3. Stability assessment based on efficiency variance in the confidence spectrum.

As shown in *Fig.* 1, the proposed aggregation model E_o shows greater stability to changes in the confidence level α and less fluctuation in the α spectrum than the cut-point model. As shown in *Fig.* 2, the cut point approach is very sensitive to the choice of $\bar{\alpha}$, which may lead to changes in the rankings, while the proposed aggregation model eliminates this arbitrary dependence and provides a more consistent evaluation framework. Finally, as shown in Fig. 3, the likelihood-based model is less interpretable due to its dependence on the parameter η. In contrast, the proposed aggregation approach simultaneously ensures stability (lower variance), interpretability, and sufficient discrimination power between DMUs.

5.4 | Comparative Results

Let E_o denote the aggregate efficiency obtained from the proposed model, $\theta_o^P(\bar{\alpha})$ be the efficiency of the point cut α, and θ_P denote the probability-based efficiency.

The comparison focuses on the following:

I. Stability of efficiency scores.

II. Sensitivity to the choice of confidence level.

III. Discriminative power between DMU_s.

The empirical results show that the proposed aggregate efficiency E_o exhibits significantly less variance in confidence levels and is free from the arbitrariness associated with the choice of A single $\bar{\alpha}$ prevents. In contrast, DEA with a point cut α provides significantly different rankings for different values of $\bar{\alpha}$, while the likelihood-based model shows less interpretability due to the choice of η.

Conflict of Interest

The authors declare no conflict of interest.

Data Availability

All data are included in the text.

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