Annals of Optimization With Applications

www.anowa.reapress.com

Ann. Optim. Appl. Vol. 2, No. 2 (2026) 125–133.

Paper Type: Original Article

Caputo-Based Numerical Modeling of Fractional Logistic Growth with Fractal Structures

Fahime Akhavan Ghassabzade[1], Mina Bagherpoorfard[2,*]

[1] Department of Mathematics, Faculty of Sciences, University of Gonabad, Gonabad, Iran; akhavan_gh@yahoo.com.
[2] Department of Mathematics, Fasa Branch, Islamic Azad University, Fasa, Iran; mi.bagherpoorfard@iau.ac.ir.

Citation:

Received: 21 September 2024 *Revised: 10 November 2024* *Accepted: 18 Januray 2025*	Akhavan Ghassabzade, F., & Bagherpoorfard, M. (2026). Caputo-based numerical modeling of fractional logistic growth with fractal structures. *Annals of Optimization With Applications*, *2*(2), 125-133.

Abstract

Transportation is one of the most important aspects of human activity, supporting various social and economic transactions. Meanwhile, in order to remain competitive, freight transportation businesses and logistics providers must deliver high-quality, dependable, and effective services. The effective design of the service network in this industry necessitates strategic and tactical decisions regarding service frequency, optimal route selection, and market share allocation among companies. In this study, we have examined the static competition between two transportation companies by utilizing a Mixed Programming -Integer Nonlinear (MINLP) model. The competition is studied by calculating the entrant's service frequency and each company's market share using a logit function. In this type of competition, the incumbent's decisions about route selection and frequency determination are known beforehand, and our goal is to maximize profits for the new market entrant. Additionally, a number of constraints have been taken into account, including route capacities, the maximum allowable frequency on each link, and penalty costs associated with the incomplete utilization of route capacities. To evaluate and validate the model, real-world data from the Iranian Road Maintenance and Transportation Organization has been employed. Furthermore, in the sensitivity analysis phase, the impacts of changing the values of important parameters on the model's outputs were evaluated. This investigation aims to improve understanding of the system's dynamics and clarify how these factors influence optimal decision-making processes.

Keywords: Fractal-fractional calculus, Caputo-type derivative, Logistic growth, Soft computing, Fractional numerical methods, Memory effect.

1 | Introduction

The logistic growth equation has long been recognized as a cornerstone in mathematical modeling of population dynamics, epidemiology, ecology, and diffusion processes in complex systems. Since its introduction by Verhulst [1], the logistic paradigm has been applied to diverse contexts ranging from

Corresponding Author: mi.bagherpoorfard@iau.ac.ir
https://doi.org/10.48314/anowa.v2i2.73

biological proliferation and tumor growth to socio-economic diffusion and ecological competition. The classical model assumes that

the growth rate depends on both the current population size and the remaining environmental capacity. While this formulation has proven remarkably versatile, it relies on a smooth and memoryless notion of time, which often fails to capture the behavior of real-world systems influenced by hereditary traits, long-term memory, or irregular temporal structures.

Classical differential equations encode local-in-time dependencies, implicitly assuming that the system's future is shaped exclusively by its present. In many natural and engineered systems, however, processes are path-dependent: past states and rates influence current trends via slow decay kernels, thresholded responses, or structured delays. As a result, convergence to equilibrium may be slower, transient regimes can be smoother or multi-phased, and the apparent capacity can evolve dynamically. Fractional calculus provides nonlocal operators that generalize differentiation to noninteger orders, embedding memory via convolution with power-law kernels. This framework has achieved notable success in modeling anomalous transport, viscoelasticity, and long-memory processes by replacing instantaneous rates with history-dependent ones [2], [3]. Yet, while fractional derivatives capture temporal memory, they typically assume a uniform time axis. Many systems also exhibit irregular or fractal-like temporal organization-bursty activity, intermittent phases, and heterogeneous clocks-where the effective "geometry" of time deviates from smooth Euclidean structure.

To address this duality, fractal-fractional derivatives combine nonlocal memory with a fractal measure of time, enabling models to simultaneously encode hereditary effects and local scaling irregularities. In the Caputo-type fractal-fractional formulation, the operator integrates a power-law memory kernel (parameterized by the fractional order $\alpha \in (0,1]$) with a Hausdorff-inspired fractal index $\beta \in (0,1]$ that reshapes the temporal metric. The resulting dynamics reflect both how strongly the past weighs on the present and how temporal increments are effectively stretched or compressed by a nonuniform geometry. This framework has been leveraged to study chaotic oscillators, anomalous diffusion, and epidemic processes with heterogeneity in contact patterns and reporting delays, illustrating its capacity to bridge local scaling anomalies with nonlocal memory in a unified calculus [4–6].

Recent advances further highlight the relevance of fractional and fractal-fractional operators in logistic-type models. Ahmed et al. [7] developed Runge-Kutta-based schemes for fractional logistic growth, demonstrating improved accuracy and stability in capturing biological dynamics. Moumen et al. [8] extended fixed-point theory to fractional logistic equations, establishing existence and uniqueness results under generalized contraction mappings. Stempin and Sumelka [9] proposed approximation techniques for variable-order Caputo derivatives, enabling flexible modeling of systems with evolving memory strength. Adak et al. [10] applied fractional calculus to epidemic models, incorporating optimization techniques to better capture disease transmission with memory effects. Finally, Armstrong and Vicol [11] investigated anomalous diffusion via fractal homogenization, illustrating how fractal structures reshape transport and dissipation, with implications for logistic-type growth in heterogeneous environments.

In this study, we apply a Caputo-type fractal-fractional derivative to the logistic growth equation and introduce a numerical method to solve it. By varying α (memory) and β (fractal geometry), we examine their combined impact on population dynamics. The resulting model offers a smooth and versatile extension of both classical and fractional approaches. Our contributions include: 1) a dimensionally consistent convolution scheme tailored to fractal-fractional operators, 2) a simplified L1-style discretization for computational efficiency, and 3) a systematic exploration of how α and β jointly modulate transient behavior and saturation delay. Beyond methodological novelty, the framework provides a prototype for simulating growth processes in biology, epidemiology, and socio-economic systems where memory and fractal-like temporal structures are indispensable.

2 | Preliminaries

2.1 | Operators

Fractional calculus generalizes the notion of differentiation and integration to noninteger orders, providing a powerful framework for modeling systems with memory and hereditary properties [2], [3]. Among the various definitions, the Caputo fractional derivative is particularly popular due to its compatibility with classical initial conditions. For a function $f: [0, T] \to \mathbb{R}$, the Caputo derivative of order $\alpha (0 < \alpha \leq 1)$ is defined as

$$ {}^{C}D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f'(\tau) d\tau, \tag{1} $$

where $\Gamma(\cdot)$ denotes the Gamma function. This operator is nonlocal, meaning that the derivative at time t depends on the entire history of $f(\tau)$ for $\tau \in [0, t]$. Such nonlocality is crucial for capturing memory effects in viscoelasticity, anomalous diffusion, and biological growth processes [12], [13].

To further incorporate irregular temporal structures, Atangana introduced the concept of fractal-fractional derivatives [4]. Let $f(t)$ be differentiable in an open interval (a, b). If f is fractal differentiable with order β, then the Caputo-type fractal-fractional derivative of order α with a power-law kernel is given by

$$ \mathrm{CFF}^{CF}D_{0^+}^{\alpha,\beta} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} D^{\beta} f(\tau) d\tau, D^{\beta} f(\tau) \equiv \frac{df}{d(\tau^{\beta})} $$

Here, β reshapes the temporal axis, effectively introducing a fractal geometry of time. This operator thus unifies two essential features: Memory (via α) and irregular scaling (via β). Recent studies have applied such operators to epidemic modeling [10], anomalous transport [11], and variable-order systems [9].

2.2 | Logistic Model (Main Equation)

The logistic growth equation is a classical model describing constrained growth under limited resources. For the state variable $X(t)$, the governing dynamics are given by

$$ \frac{dX}{dt} = \rho X(t) \left(1 - \frac{X(t)}{K}\right), \tag{1a} $$

(Classical integer-order)

$$ C^{C}D_{0^+}^{\alpha} X(t) = \rho X(t) \left(1 - \frac{X(t)}{K}\right), 0 < \alpha \leq 1, \tag{1b} $$

$$ {}^{CFF}D_{0^+}^{\alpha,\beta} X(t) = \rho X(t) \left(1 - \frac{X(t)}{K}\right), 0 < \alpha, \beta \leq 1, \tag{1c} $$

(Fractional Caputo)

with initial condition $X(0) = X_0 \in (0, K)$ and parameters $\rho > 0$ (intrinsic growth rate) and $K > 0$ (carrying capacity).

Eq. (1a) represents the classical logistic law introduced by Verhulst [1]. *Eq. (1b)* extends the model by incorporating fractional memory through the Caputo derivative, allowing the system to reflect hereditary effects and delayed saturation [7], [8]. Finally, *Eq. (1c)* introduces both fractional memory and fractal temporal geometry, enabling the model to capture smoother transients and irregular convergence patterns observed in real-world processes [14].

3 | Discussion

The inclusion of fractional and fractal-fractional operators in logistic dynamics provides several advantages:

I. Memory effects: Fractional derivatives allow the system to retain influence from past states, crucial for biological and epidemiological modeling.

II. Fractal time geometry: The fractal index β modifies the effective pacing of time, capturing bursty or irregular growth patterns.

III. Numerical schemes: Recent work has developed Runge-Kutta and convolution-based discretizations for fractional logistic equations, ensuring stability and accuracy [7].

IV. Applications: Models with α and β parameters have been applied to epidemic spread, tumor growth, ecological succession, and socio-economic diffusion [8], [10].

Thus, the preliminaries establish the mathematical foundation for our study: the logistic equation under classical, fractional, and fractal-fractional formulations. This framework enables systematic exploration of how memory and fractal geometry jointly shape growth dynamics, providing a versatile prototype for complex systems.

3 | Existence and Uniqueness Analysis

3.1 | Problem Formulation

Consider the fractal-fractional logistic equation

$$\mathrm{CFF}^{\mathrm{CF}}D_{0^+}^{\alpha,\beta}X(t) = \rho X(t)\left(1 - \frac{X(t)}{K}\right), t > 0. \quad \textbf{(2)}$$

with initial condition $X(0) = X_0$, where $0 < \alpha, \beta \leq 1$ and $\rho, K > 0$. The fractal-fractional derivative operator is defined as [15], [16].

$$\mathrm{CFF}^{\alpha,\beta}D_{0^+}^{\alpha,\beta}(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}D^{\beta}f(\tau)d\tau, D^{\beta}f(\tau) \equiv \frac{df}{d(\tau^{\beta})}. \quad \textbf{(3)}$$

Here, β introduces a fractal scaling of the temporal axis, while α represents the fractional order of differentiation.

3.2 | Equivalent Integral Formulation

To analyze the existence and uniqueness of solutions, we transform the initial value problem into an equivalent integral equation. Applying the inverse operator of ${}^{\mathrm{CFF}}D_{0^+}^{\alpha,\beta}$, we obtain [4]:

Lemma 1. The initial value problem for *Eq. (2)* is equivalent to the nonlinear Volterra integral equation

$$X(t) = X_0 + \int_0^t K(t,\tau)F\big(X(\tau)\big)d\tau, t \in [0,T], \quad \textbf{(4)}$$

where $F(X) = \rho X(1 - X/K)$ and the kernel $K(t,\tau)$ is given by

$$K(t,\tau) = \frac{\beta}{\Gamma(\alpha)}\left[t^{\beta-1}(t-\tau)^{\alpha-1} - (\beta-1)\int_0^{t-\tau}(u+\tau)^{\beta-2}u^{\alpha-1}du\right]. \quad \textbf{(5)}$$

Proof: The proof follows from the definition of the inverse fractal-fractional operator [4] and properties of fractional integrals [3].

Remark 1. The kernel (5) exhibits a weak singularity at $\tau = t$ due to the term $(t-\tau)^{\alpha-1}$ with $0 < \alpha < 1$. To ensure integrability, we assume the condition

$$\alpha + \beta > 1. \quad \textbf{(6)}$$

Which guarantees that $K(t,\tau) \in L^1([0,T] \times [0,T])$ based on properties of fractional integrals [3], [17]. For $\beta = 1$, the kernel reduces to a form associated with non-singular fractional operators [18].

3.3 | Local Existence and Uniqueness

Let $C[0,T]$ denote the space of continuous functions on $[0,T]$ equipped with the supremum norm $\|X\|_\infty = \sup_{t\in[0,T]} |X(t)|$. Define the operator $\mathcal{T}: C[0,T] \to C[0,T]$ by

$$(\mathcal{T}X)(t) = X_0 + \int_0^t K(t,\tau)F(X(\tau))d\tau \tag{7}$$

Fixed points of $\mathcal{T}$ correspond to solutions of *Eq. (4)*. The nonlinearity F is locally Lipschitz continuous. For the logistic nonlinearity, for any bounded set $B \subset \mathbb{R}$, there exists a constant L_B such that

$$|F(x) - F(y)| \leq L_B|x - y|, \forall\, x, y \in B, \tag{8}$$

Where

$$L_B = \rho\left(1 + 2\sup_{x\in B}|x|/K\right)$$

Theorem 1 (local existence and uniqueness). Assume *Condition (6)* holds. Then, for sufficiently small $T > 0$, the operator $\mathcal{T}$ is a contraction on $C[0,T]$. Consequently, the fractal-fractional logistic equation admits a unique continuous solution on $[0,T]$.

Proof: Under *Condition (6)*, the kernel $K(t,\tau)$ is bounded, i.e., there exists $M > 0$ such that $|K(t,\tau)| \leq M$ for all $t, \tau \in [0,T]$ [17]. Choose a ball $B_R = \{X \in C[0,T]: \|X - X_0\|_\infty \leq R\}$ and let L_B be the Lipschitz constant of F on $[X_0 - R, X_0 + R]$. For any $X, Y \in B_R$, we have

$$\begin{aligned}\|\mathcal{T}X - \mathcal{T}Y\|_\infty &\leq \sup_{t\in[0,T]} \int_0^t |K(t,\tau)||F(X(\tau)) - F(Y(\tau))|d\tau \\ &\leq ML_B \int_0^t \|X - Y\|_\infty d\tau \\ &\leq ML_B T\|X - Y\|_\infty\end{aligned}$$

Select T such that $\theta = ML_BT < 1$. Then $\mathcal{T}$ is a contraction on B_R. Moreover, for sufficiently small $T, \mathcal{T}$ maps B_R into itself. By the Banach fixed-point theorem, $\mathcal{T}$ has a unique fixed point in B_R, which is the desired solution. The Banach fixed-point theorem is a standard result in functional analysis (see, e.g., the approach in [2]).

3.4 | Global Existence

The local solution can be extended to a maximal interval of existence $[0, t_{max})$. Using standard continuation arguments for integral equations, the solution can be extended as long as it remains bounded.

Theorem 2 (global existence). Under *Condition (6)* and $X_0 \geq 0$, the unique solution guaranteed by *Theorem 1* exists for all $t \geq 0$ and satisfies $0 \leq X(t) \leq \max\{X_0, K\}$.

Proof: First, note that $F(0) = 0$. Using comparison principles for fractional equations [8], one can show that $X(t) \geq 0$ for all $t \geq 0$ when $X_0 \geq 0$.

For boundedness, observe that the logistic nonlinearity ensures bounded growth. If $X(t)$ were to exceed $\max\{X_0, K\}$, then $F(X(t))$ would become negative, causing $X(t)$ to decrease. This self-limiting behavior prevents blow-up in finite time.

Since the solution remains bounded, by standard extension theorems for Volterra integral equations, the maximal interval of existence is $[0, \infty)$. This extension argument follows principles similar to those for ordinary differential equations as discussed in [2].

3.5 | Remarks and Discussion

I. The condition $\alpha + \beta > 1$ is sufficient for integrability of the kernel and can be relaxed by working in weighted function spaces, but for most practical applications in biological modeling [12], this condition is reasonable.

II. The fractal parameter β modifies the kernel's structure, influencing the solution's regularity and asymptotic behavior. For $\beta = 1$, we recover fractional models studied in [13], [19].

III. The existence and uniqueness results provide a solid foundation for numerical approximation methods such as those discussed in [7], [9].

IV. The proof technique is general and can be extended to other fractal-fractional models in epidemiology [10], [14] and biological systems [5], [8].

These results establish that the fractal-fractional logistic model is well-posed, ensuring its suitability for further analytical and numerical investigations of complex growth phenomena.

4 | Numerical Results

4.1 | Numerical Discretization

For the numerical solution, we adopt a Caputo-type fractal-fractional derivative based on the Hausdorff fractal measure. Let $t_n = t_0 + nh$, where $h > 0$ is the uniform time step. The fractal derivative is approximated as

$$D^{\beta}X(t_k) \approx \frac{X_{k+1} - X_k}{((k+1)^{\beta} - k^{\beta})h^{\beta}}.$$

By integrating the kernel of the Caputo operator exactly, the following discretization is obtained:

$$\mathrm{CFFD}_{0+}^{\alpha,\beta}X(t_n) \approx \frac{1}{\Gamma(2-\alpha)}h^{-(\alpha+\beta-1)}\sum_{k=0}^{n-1}[(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}]\big((k+1)^{\beta} - k^{\beta}\big)(X_{k+1} - X_k).$$

4.2 | Numerical Experiments

To study the influence of both fractional memory (α) and fractal scaling (β), we tested three representative cases:

Table 1. Representative cases used to investigate the effects of fractional memory (α) and fractal scaling (β).

Case	α	β	Description
1	1.0	1.0	Classical logistic model
2	0.8	1.0	Fractional memory only
3	0.8	0.8	Combined fractional-fractal case

Fig. 1 illustrates the time evolution of the population $X(t)$ for three parameter configurations. When $\alpha = 1$ and $\beta = 1$, the system follows the classical logistic growth pattern, where the population initially grows exponentially and then stabilizes as it reaches the carrying capacity $K = 1$. This behavior represents an idealized process without memory or structural complexity, where changes in population depend only on the current state.

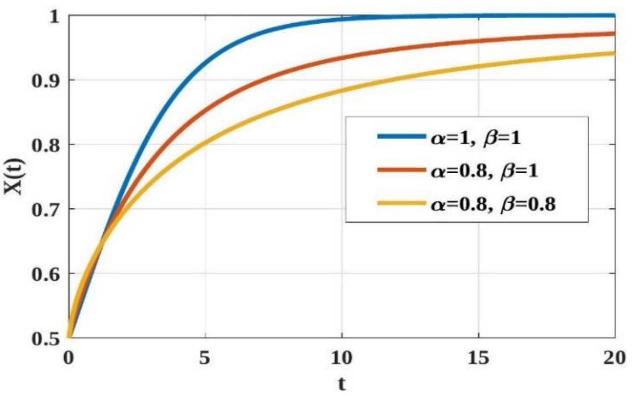

Fig. 1. Comparison of the classical, fractional, and fractal-fractional logistic models.

The plots illustrate the temporal evolution of the normalized population $X(t)$ for three different derivative types: 1) the classical integer-order case $(\alpha, \beta) = (1,1)$, 2) the fractional Caputo derivative $(\alpha, \beta) = (0.8,1)$, and 3) the Caputo-type fractal-fractional derivative $(\alpha, \beta) = (0.8,0.8)$. The decrease of α and β produces a slower growth rate and a delayed saturation, reflecting the memory and irregular temporal geometry of the system.

For $\alpha = 0.8$ and $\beta = 1$, the fractional derivative introduces a memory component into the system. In this case, the population growth depends not only on its present value but also on its past evolution. As a result, the increase in $X(t)$ becomes more gradual, and the system requires a longer time to reach equilibrium. This behavior reflects a realistic biological scenario where the effects of past states, such as resource depletion or delayed responses, influence the rate of population change.

When both parameters take fractional and fractal values, namely $\alpha = 0.8$ and $\beta = 0.8$, the combined influence of memory and geometric complexity becomes evident. The fractal time scaling modifies the effective pace of temporal evolution, while the fractional order maintains the system's dependence on its history. Together, these effects generate the slowest approach to saturation, representing a population evolving in an environment with both long-term memory and irregular structural constraints. Quantitatively, the time required for the population to reach 90% of the carrying capacity K (denoted by $t_{0.9}$) is summarized in *Table 1*.

As seen, $t_{0.9}$ increases as the fractional and fractal effects become more pronounced, confirming the slower convergence observed in the numerical curves. Quantitatively, the time required for the population to reach 90% of the carrying capacity K (denoted by $t_{0.9}$) is summarized in *Table 1*. As seen, $t_{0.9}$ increases as the fractional and fractal effects become more pronounced, confirming the slower convergence observed in the numerical curves. The numerical experiments demonstrate that:

Table 1. Time required for the solution to reach 90% of the carrying capacity ($t_{0.9}$) for different parameter configurations.

	$(\alpha, \beta) = (1, 1)$	$(0.8, 1)$	$(0.8, 0.8)$
$t_{0.9}$	4.1	6.8	9.5

I. The classical logistic growth is recovered when $\alpha = \beta = 1$.

II. Decreasing α introduces memory effects that slow the evolution.

III. Decreasing β further modifies the timescale due to the fractal structure of time.

Overall, the results confirm that the combined fractional operator is a powerful framework for modeling nonlinear dynamical systems that exhibit both memory and fractal temporal scaling properties.

5 | Conclusion

This study compared classic, fractional, and fractal-fractional logistic models. Results indicate that incorporating fractional and fractal parameters enhances the models' capacity to reflect systems with memory effects and irregular temporal dynamics. The fractal-fractional model specifically shows slower convergence and more intricate dynamical behaviour relative to the traditional model. Future work will focus on validating these findings with real data and refining parameter estimation techniques to increase accuracy and efficiency. To address this duality, fractal-fractional derivatives combine nonlocal memory with a fractal measure of time, enabling models to simultaneously encode hereditary effects and local scaling irregularities.

In the Caputo-type fractal-fractional formulation, the operator integrates a power-law memory kernel (parameterized by the fractional order $\alpha \in (0,1])$ with a Hausdorff-inspired fractal index $\beta \in (0,1]$ that reshapes the temporal metric. The resulting dynamics reflect both how strongly the past weighs on the present and how temporal increments are effectively stretched or compressed by a nonuniform geometry. This framework has been leveraged to study chaotic oscillators, anomalous diffusion, and epidemic processes with heterogeneity in contact patterns and reporting delays, illustrating its capacity to bridge local scaling anomalies with nonlocal memory in a unified calculus [5], [9], [16].

Conflict of Interest

The authors declare that they have no competing interests.

Data Availability

The data supporting the findings of this study are available within the article.

Funding

This research did not receive any external funding.

References

[1] Verhulst., P. F. (1838). Notice sur la loi que la population suit dans son accroissement. *Correspondence mathematique et physique*, *10*, 113–129. https://cir.nii.ac.jp/crid/1570009749935841536

[2] Diethelm, K., & Ford, N. J. (2010). The analysis of fractional differential equations. *Lecture notes in mathematics*, *2004*. https://doi.org/10.1007/978-3-642-14574-2

[3] Podlubny, I. (1999). *Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. Academic Press. https://books.google.com/books/about/Fractional_Differential_Equations.html?id=K5FdXohLto0C

[4] Atangana, A., & Akgül, A. (2020). Analysis of new trends of fractional differential equations. In *Fractional order analysis* (pp. 91–111). John Wiley & Sons. https://doi.org/10.1002/9781119654223.ch3

[5] Karaagac, B., Owolabi, K. M., & Pindza, E. (2023). A computational technique for the Caputo fractal-fractional diabetes mellitus model without genetic factors. *International journal of dynamics and control*, *11*(5), 2161–2178. https://doi.org/10.1007/s40435-023-01131-7

[6] Sun, H., Zhang, Y., Baleanu, D., Chen, W., & Chen, Y. (2018). A new collection of real world applications of fractional calculus in science and engineering. *Communications in nonlinear science and numerical simulation*, *64*, 213–231. https://doi.org/10.1016/j.cnsns.2018.04.019

[7] Ahmed, S., Ullah, S., Riaz, M., Ahmed, W., & Sheikh, M. (2025). Numerical analysis of fractional logistic growth models using computational methods with MATLAB codes. *International journal of applied and computational mathematics*, *11*(5), 198. https://doi.org/10.1007/s40819-025-02003-8

[8] Moumen, A., Saleh, H. N., Albala, H., Aldwoah, K., Saber, H., Hassan, E. I., & Hassan, T. S. (2025). On polynomial φ-contractions with applications to fractional logistic growth equations. *Fractal and fractional*, *9*(6), 366. https://doi.org/10.3390/fractalfract9060366

[9] Stempin, P., & Sumelka, W. (2025). Approximation of fractional caputo derivative of variable order and variable terminals with application to initial/boundary value problems. *Fractal and fractional*, *9*(5), 269. https://doi.org/10.3390/fractalfract9050269

[10] Adak, S., Barman, S., Jana, S., Majee, S., & Kar, T. K. (2025). Modelling and analysis of a fractional-order epidemic model incorporating genetic algorithm-based optimization. *Journal of applied mathematics and computing*, *71*(1), 901–925. https://doi.org/10.1007/s12190-024-02224-y

[11] Armstrong, S., & Vicol, V. (2025). Anomalous diffusion by fractal homogenization. *Annals of pde*, *11*(1), 2. https://doi.org/10.1007/s40818-024-00189-6

[12] Alinei-Poiana, T., Dulf, E. H., & Kovacs, L. (2023). Fractional calculus in mathematical oncology. *Scientific reports*, *13*(1), 10083. https://doi.org/10.1038/s41598-023-37196-9

[13] Tarasov, V. E. (2010). *Fractional dynamics: Applications of fractional calculus to dynamics of particles, fields and media*. Springer Berlin, Heidelberg. https://doi.org/10.1007/978-3-642-14003-7

[14] Gunasekar, T., Manikandan, S., Haque, S., Suba, M., & Mlaiki, N. (2025). Fractal-fractional mathematical modeling of monkeypox disease and analysis of its Ulam–Hyers stability. *Boundary value problems*, *2025*(1), 20. https://doi.org/10.1186/s13661-025-02013-x

[15] Atangana, A. (2017). Fractal-fractional differentiation and integration: Connecting fractal calculus and fractional calculus to predict complex system. *Chaos, solitons & fractals*, *102*, 396–406. https://doi.org/10.1016/j.chaos.2017.04.027

[16] Atangana, A., & Baleanu, D. (2016). New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model. *Thermal science*, *20*(3), 763–769. https://doi.org/10.2298/TSCI160111018A

[17] Samko, S. G. (1993). *Fractional integrals and derivatives: Theory and applications*. Gordan and Breach Science. Gordon & Breach Science Pub.

[18] Caputo, M., & Fabrizio, M. (2015). A new definition of fractional derivative without singular kernel. *Progress in fractional differentiation & applications*, *1*(2), 73–85. http://dx.doi.org/10.12785/pfda/010201

[19] El-Sayed, A. M. A., El-Mesiry, A. E. M., & El-Saka, H. A. A. (2007). On the fractional-order logistic equation. *Applied mathematics letters*, *20*(7), 817–823. https://doi.org/10.1016/j.aml.2006.08.013