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Inverse Cost Efficiency Data Envelopment Analysis with a Network Structure in the Presence of Fuzzy Data

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Abstract

The petrochemical industry is one of the most influential sectors in the economies of oil-producing countries, particularly Iran, as well as the global economy. To make informed managerial decisions in this field, managers must not only evaluate the past performance of organizations and identify sources of inefficiency, but also consider existing operational constraints. Subsequently, strategic objectives should be established in accordance with the macro-level policies of the industry, enabling organizations to compete effectively in the global market and move toward sustainable future development. Data Envelopment Analysis (DEA) is a powerful retrospective performance evaluation tool, whereas Inverse Data Envelopment Analysis (Inverse DEA) is a forward-looking approach capable of estimating the required input or output values while preserving efficiency. The integration of these two powerful methodologies enables managers to develop flexible and informed plans for future decision-making. Furthermore, data encountered in real-world applications are often characterized by uncertainty and imprecision; therefore, it is essential to extend conventional network DEA models to accommodate fuzzy data in order to obtain more realistic and reliable results. This study proposes a novel fuzzy arithmetic-based inverse Network Data Envelopment Analysis (NDEA) model that estimates the required input values while maintaining both the overall process efficiency and cost efficiency at constant levels. Meanwhile, output values are adjusted according to managerial preferences and organizational objectives. The proposed model is applied to a manufacturing workshop in the oil and petrochemical industry, and the results demonstrate its high effectiveness and practical applicability.

Keywords: Two-stage network data envelopment analysis, Inverse data envelopment analysis, Cost efficiency, Fuzzy data.

1 | Introduction

In today's rapidly evolving environment, informed decision-making has become a fundamental requirement for senior managers seeking to develop effective strategies for future planning. A key prerequisite for such planning is the evaluation of past organizational performance and the identification of sources of inefficiency.

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Among the various performance evaluation techniques, Data Envelopment Analysis (DEA) has emerged as one of the most effective non-parametric methods for assessing the relative efficiency of homogeneous Decision-Making Units (DMUs). Based on linear programming, DEA constructs a Production Possibility Set (PPS) according to a set of axioms, where the frontier of the PPS represents an approximation of the production function. A DMU located on the efficient frontier is regarded as efficient and receives an efficiency score of one. In contrast, inefficient units obtain efficiency scores below one and can be projected onto the efficient frontier to determine appropriate performance benchmarks.

The foundations of DEA were established by Farrell [1], who introduced a model for measuring the efficiency of units with multiple inputs and a single output. Charnes and Cooper [2] later extended Farrell's work to accommodate multiple inputs and outputs, introducing the CCR model, while Banker et al. [3] subsequently developed the BCC model. Since then, numerous radial and non-radial DEA models have been proposed, enabling more accurate evaluations of relative efficiency and more precise identification of the sources of inefficiency within DMUs.

In practice, many governmental and non-governmental organizations, such as banks, educational institutions, and manufacturing firms, operate through two-stage or multi-stage production processes organized in either series or parallel network structures. Conventional DEA models evaluate these organizations as black-box systems, considering only the initial inputs and final outputs while ignoring the intermediate measures generated throughout the production process. Consequently, these models are unable to reveal the sources of inefficiency within individual subprocesses. To overcome this limitation, Network Data Envelopment Analysis (NDEA) has been developed for evaluating multi-stage production systems. In particular, the two-stage DEA model forms the basis of network DEA, where the outputs of the first stage become the intermediate measures serving as inputs to the second stage, which ultimately produces the final outputs.

Although mathematical models are intended to represent real-world systems, the data encountered in practice are rarely precise and deterministic. Instead, they are often characterized by uncertainty and vagueness, making fuzzy set theory, first introduced by Zadeh [4], an effective framework for handling such information. The incorporation of fuzzy numbers into DEA has significantly enhanced its applicability in industrial and managerial decision-making. However, solving fuzzy DEA models remains a challenging task, and several approaches have been proposed, including the α-cut approach, fuzzy number ranking methods, the possibility approach, and fuzzy arithmetic. Among these, the fuzzy arithmetic approach has attracted considerable attention because it preserves the original fuzzy information without requiring defuzzification. Wang et al. [5] introduced a fuzzy arithmetic framework for fractional DEA models in which triangular fuzzy numbers represent inputs and outputs. Their approach transforms each fuzzy fractional model into three linear programming models corresponding to the lower, most likely, and upper bounds of efficiency. Subsequently, Baradwaj et al. [6] refined this methodology, while Kachoui et al. [7] proposed a fuzzy arithmetic-based model for evaluating the efficiency of two-stage processes. Furthermore, Saeedi et al. [8], [9] developed a non-radial weighted additive model for decomposing the efficiency of two-stage processes with fuzzy data in the presence of undesirable intermediate measures.

Another important concept in DEA is cost efficiency, which evaluates the capability of DMUs to utilize resources optimally with respect to their associated costs. Cost efficiency reflects the extent to which a DMU can minimize its costs while maintaining a given level of outputs and is defined as the product of Technical Efficiency (TE) and Allocative Efficiency (AE). This measure not only evaluates whether inputs are used efficiently but also determines whether they are combined in an economically optimal manner according to their prices. Consequently, cost efficiency analysis provides managers with a powerful tool for improving resource allocation and identifying avoidable costs. The concept was originally introduced by Farrell [1], who argued that an efficient DMU should simultaneously maximize production from available inputs and select the least-cost combination of those inputs.

While conventional DEA is primarily retrospective, Inverse Data Envelopment Analysis (IDEA) extends DEA into a forward-looking decision-support framework by estimating future operational activities following potential changes while preserving the current efficiency level. In practice, managers frequently encounter situations in which resource limitations, investment decisions, or strategic planning require adjustments to inputs or outputs without altering the relative efficiency of the organization. Consequently, inverse DEA addresses two important questions: 1) how much additional output should be produced when inputs increase while maintaining the existing efficiency level, and 2) how much additional input is required to achieve a desired increase in outputs without changing efficiency. Since its introduction by Wei et al. [10], inverse DEA has attracted considerable research interest, with significant contributions by Yan [11], Jahanshahloo et al. [12], [13], Hadi-Vencheh and Foroughi [14], and others. More recently, Shiri Daryani et al. [15], [16] applied inverse DEA within a two-stage network structure using a multiplicative approach to evaluate cost efficiency.

Motivated by these developments, this study proposes a novel DEA framework for evaluating TE and cost efficiency in two-stage network systems with fuzzy data. Unlike existing approaches, the proposed method employs the fuzzy arithmetic approach to transform the fuzzy optimization problem directly into three linear programming models corresponding to the lower, most likely, and upper bounds, thereby eliminating the need for defuzzification while preserving the transparency of the original data. Furthermore, the proposed framework introduces computationally efficient linear programming models for estimating outputs through Inverse DEA, while explicitly incorporating managerial preferences by assigning appropriate weights during the estimation process. Finally, the applicability of the proposed methodology is demonstrated through a real-world case study in the oil and petrochemical equipment industry, confirming its effectiveness and practical value.

The remainder of this paper is organized as follows. Section 2 presents the independent and relational network DEA models. Section 3 introduces the basic concepts of fuzzy data and fuzzy arithmetic operations. Section 4 reviews the concept of cost efficiency and the classical models developed for its evaluation. Section 5 discusses the principles of IDEA and the fundamental models for estimating inputs and outputs. Section 6 presents the proposed methodology, which evaluates the technical and cost efficiency of two-stage networks with fuzzy inputs, outputs, and intermediate measures through three linear programming models while simultaneously estimating output levels based on managerial preferences. Finally, Section 7 reports the implementation of the proposed model using data collected from a manufacturing workshop in the oil and petrochemical equipment industry and discusses the obtained results.

2 | Efficiency Model of DEA in a Two-Stage Process

Assume that there are n DMUs, denoted by $DMU_j(j,=,1 \dots n)$, operating under a two-stage series structure. In the first subprocess, the inputs $x_{ij}(i,=,1 \dots m)$ are consumed to produce the intermediate outputs $z_{pj}(p,=,1 \dots q)$, which are referred to as intermediate measures. These intermediate measures are subsequently consumed in the second subprocess to generate the final outputs $y_{rj}(r,=,1 \dots s)$.

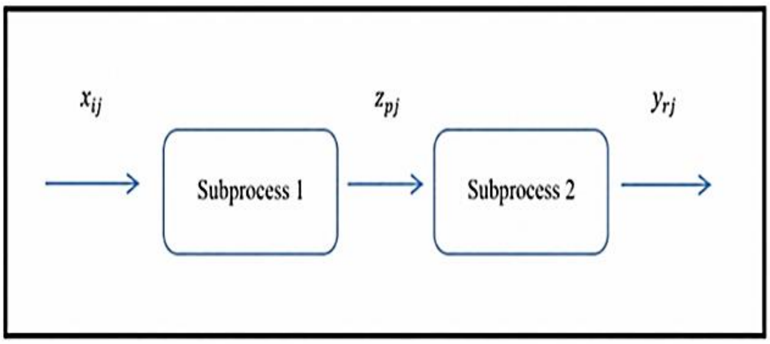

Fig. 1. Two-stage process with a structure.

2.1 | Conventional DEA Efficiency Model for a Two-Stage Process

Conventional DEA models, also referred to as the independent approach, evaluate only the overall efficiency of a two-stage process while disregarding the intermediate measures between the two stages. Consequently, these models are unable to identify the sources of inefficiency within individual subprocesses. Charnes et al. [2] proposed the following model:

$$\begin{aligned}
&E_o = \max \sum_{r=1}^{s} u_r y_{ro},\\
&\text{s.t.}\quad \sum_{i=1}^{m} v_i x_{io} = 1,\\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \qquad j = 1,\dots,n,\\
&v_i,, u_r \ge 0, \qquad i = 1,\dots,m\ ,\ r = 1,\dots,s.
\end{aligned} \tag{1}$$

The optimal objective value of *Model (1)* is denoted by (E_o^*, v_i^*, u_r^*).

2.2 | Relational DEA Efficiency Model for a Two-Stage Process

Under the relational approach, the intermediate measures are explicitly incorporated into the evaluation of the overall efficiency of the two-stage process. The model for assessing the overall efficiency of a two-stage process under the Constant Returns to Scale (CRS) assumption was proposed by Kao and Hwang [17] as follows:

$$\begin{aligned}
&E_o = \max \sum_{r=1}^{s} u_r y_{ro},\\
&\text{s.t.}\quad \sum_{i=1}^{m} v_i x_{io} = 1,\\
&\quad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \qquad j = 1,\dots,n,\\
&\quad \sum_{p=1}^{q} w_p z_{pj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \qquad j = 1,\dots,n,\\
&\quad \sum_{p=1}^{q} w_p z_{pj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \qquad j = 1,\dots,n,\\
&\quad v_i, w_p, u_r \ge 0, \qquad i = 1,\dots,m\ , p = 1,\dots,q,\ r = 1,\dots,s.
\end{aligned} \tag{2}$$

The envelopment form of *Model (2)* is given as follows:

$$\begin{aligned}
&\min\ \theta\\
&\text{s.t.}\quad \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{ij}\ , \qquad i = 1,\dots,m,\\
&\quad \sum_{j=1}^{n} \lambda_j z_{pj} \ge z_{po}\ , \qquad i = 1,\dots,m,\\
&\quad \sum_{j=1}^{n} \mu_j z_{pj} \le z_{po}\ , \qquad i = 1,\dots,m,\\
&\quad \sum_{j=1}^{n} \mu_j y_{rj} \ge y_{ro}, \qquad r = 1,\dots,s,\\
&\quad \lambda_j \ge 0, \mu_j \ge 0\ , \qquad j = 1,\dots,n.
\end{aligned} \tag{3}$$

3 | Fuzzy Data

In real-world applications, input and output data are not always precise and deterministic; rather, they are often characterized by uncertainty and imprecision. Consequently, one of the major challenges associated with classical DEA models is their ability to effectively incorporate fuzzy data into the efficiency evaluation process.

3.1 | Fundamental Concepts and Definitions of Fuzzy Numbers

Definition 1 . A triangular fuzzy number $\tilde{A}$ is represented as [4], [18]:

$$\tilde{A} = \left(a^l, a^m, a^u\right).$$

Whose membership function is defined as

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a^l}{a^m - a^l}, & a^l \le x \le a^m, \\ \dfrac{a^u - x}{a^u - a^m}, & a^m \le x \le a^u. \end{cases} \tag{4}$$

Definition 2. A triangular fuzzy number:

$$\tilde{a} = \left(a^l, a^m, a^u\right).$$

is said to be positive if

$$a^l > 0.$$

Definition 3. Consider two positive triangular fuzzy numbers:

$$\tilde{a} = \left(a^l, a^m, a^u\right) \text{ and } \tilde{b} = \left(b^l, b^m, b^u\right).$$

Their comparison is defined as follows:

$$\tilde{a} \le \tilde{b} \to \left(a^l \le b^l, a^m \le b^m, a^u \le b^u\right).$$

The basic fuzzy arithmetic operations for two positive triangular fuzzy numbers, $A = \left(a^l, a^m, a^u\right)$ and $B = \left(b^l, b^m, b^u\right)$, are defined as follows:

$$\begin{gathered} \tilde{A} \oplus \tilde{B} = \left(a^l + b^l, a^m + b^m, a^u + b^u\right), \\ \tilde{A} \ominus \tilde{B} = \left(a^l - b^u, a^m - b^m, a^u - b^l\right), \\ \tilde{A} \otimes \tilde{B} = \left(a^l \times b^l, a^m \times b^m, a^u \times b^u\right), \\ k\tilde{A} = \begin{cases} \left(ka^l, ka^m, ka^u\right) & k > 0 \\ \left(ka^u, ka^m, ka^l\right) & k < 0 \end{cases}, \quad k \in \mathbb{R}, \\ \frac{\tilde{A}}{\tilde{B}} = \frac{\left(a^l, a^m, a^u\right)}{\left(b^l, b^m, b^u\right)} = \left(\frac{a^l}{b^u}, \frac{a^m}{b^m}, \frac{a^u}{b^l}\right), \quad \tilde{A}, \tilde{B} > \tilde{0}. \end{gathered} \tag{5}$$

4 | Cost Efficiency

Assume that $c_i \in \mathbb{R}(i, =, 1 \ldots m)$ denotes the unit cost (weight) associated with input $x_i(i, =, 1 \ldots m)$, and that

$c^T x^o = \sum_{i=1}^{m} c_i\, x_{io}$.

Represents the observed cost of the DMU_o under evaluation. To determine the cost efficiency of the evaluated DMU, the following model is first solved [1]:

$$
\begin{aligned}
& c^t X^* = \min \ \sum_{i=1}^{m} c_i x_i, \\
& \text{s.t.} \ \ \sum_{j=1}^{n} \lambda_j x_{ij} \le x_{io}, && i = 1,\dots,m, \\
& \lambda_j y_{rj} \ge y_{ro}, && r = 1,\dots,s, \\
& \lambda_j \ge 0, && j = 1,\dots,n.
\end{aligned} \tag{6}
$$

Let (x^*, λ^*) denote the optimal solution of *Model (6)*. In fact, the objective of this model is to determine the minimum input vector x^* capable of producing at least the same level of outputs as DMU_o. Therefore, the minimum attainable cost obtained from the model is

$$
c^T x^* = \sum_{i=1}^{m} c_i\, x_i^*.
$$

Accordingly, the cost efficiency of the DMU is defined as follows:

$$
CE_o = \frac{c^t X^*}{c^t X_o} = \frac{\sum_{i=1}^{m} c_i x_i^*}{\sum_{i=1}^{m} c_i x_{io}}. \tag{7}
$$

Definition 4. A DMU is said to be cost efficient if its cost efficiency score is equal to one. Otherwise, it is considered cost inefficient, and its cost efficiency score is less than one.

4.1 | DEA Cost Efficiency Model for a Two-Stage Process

The envelopment form of the two-stage DEA cost efficiency model proposed by Shiri Daryani et al. [16] can be written as follows:

$$
\begin{aligned}
& c^t x^* = \min \ \sum_{i=1}^{m} c_i x_i, \\
& \text{s.t.} \ \ \sum_{j=1}^{n} \lambda_j x_{ij} \le x_{io}, && i = 1,\dots,m, \\
& \sum_{j=1}^{n} (\lambda_j - \mu_j) z_{pj} \ge z_{po}, && p = 1,\dots,q, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro}, && r = 1,\dots,s, \\
& \lambda_j \ge 0, \quad \mu_j \ge 0, && j = 1,\dots,n.
\end{aligned} \tag{8}
$$

If (x^*, λ^*, μ^*) denotes the optimal objective value of the above model, then:

$$
CE_o = \frac{c^t x^*}{c^t x_o} = \frac{\sum_{i=1}^{m} c_i x_i^*}{\sum_{i=1}^{m} c_i x_{io}}. \tag{9}
$$

5 | Inverse Data Envelopment Analysis

Suppose that the output of the DMU_o under evaluation, DMU_o, changes from y_{ro} to:

$$\beta_{ro} = y_{ro} + \Delta y_{ro}.$$

The question then arises: To what extent should the input x_{io} be changed? In other words, what should the new input level,

$$\alpha_{io} = x_{io} + \Delta x_{io}.$$

Be in order to preserve the efficiency of the unit? IDEA addresses this question. Wei et al. [10] were the first to propose the following model:

$$\begin{aligned}
&\min\ (\alpha_1, \alpha_2, \ldots, \alpha_m),\\
&\text{s.t.}\ \sum_{j=1}^{n} \lambda_j x_{ij} \le \alpha_{io}, && i = 1,\ldots,m,\\
&\sum_{j=1}^{n} \lambda_j y_{rj} \ge \beta_{ro}, && r = 1,\ldots,s,\\
&\lambda_j \ge 0, && j = 1,\ldots,n.
\end{aligned} \tag{10}$$

The above model is a multi-objective optimization model, for which various solution approaches have been developed. One of the simplest and most widely used methods is the weighted-sum method.

$$w_1\alpha_1 + w_2\alpha_2 + \ldots + w_m\alpha_m = \sum_{i=1}^{m} w_i\alpha_i. \tag{11}$$

The weights can also be selected based on the manager's preferences. In this study, equal weights are assigned to all objectives; that is $\sum_{i=1}^{m} w_i = 1$.

6 | Proposed Approach

6.1 | Relational DEA Efficiency Model for a Two-Stage Network with Fuzzy Data (Input-Oriented)

The envelopment form of the input-oriented relational DEA model for a two-stage network, originally proposed by Kao and Hwang [17] is extended to accommodate fuzzy data, resulting in *Model (12).*

$$\begin{aligned}
&\min\ \tilde{\theta},\\
&\text{s.t.}\ \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \le \tilde{\theta}\tilde{x}_{ij}, && i = 1,\ldots,m,\\
&\sum_{j=1}^{n} \lambda_j \tilde{z}_{pj} \ge \tilde{z}_{pj}, && p = 1,\ldots,q,\\
&\sum_{j=1}^{n} \mu_j \tilde{z}_{pj} \le \tilde{z}_{pj}, && p = 1,\ldots,q,\\
&\sum_{j=1}^{n} \mu_j \tilde{y}_{rj} \ge \tilde{y}_{ro}, && r = 1,\ldots,s,\\
&\lambda_j \ge 0,\ \mu_j \ge 0, && j = 1,\ldots,n.
\end{aligned} \tag{12}$$

Let $\left(\tilde{\theta}^*, \lambda^* \mu^*\right)$ denote the optimal objective value of the above model.

Assume that the fuzzy input data, output data, and intermediate measures are represented by triangular fuzzy numbers as follows:

$$\tilde{x}_{ij} = \left(x_{ij}^{l}, x_{ij}^{m}, x_{ij}^{u}\right),\ \tilde{y}_{rjk} = \left(y_{rj}^{l}, y_{rj}^{m}, y_{rj}^{u}\right),\ \tilde{z}_{ijk} = \left(z_{pj}^{l}, z_{pj}^{m}, z_{pj}^{u}\right). \quad \textbf{(13)}$$

$$\begin{aligned}
&\theta_o^{l^*} = \min\ \theta_o^l, \\
&\text{s.t.}\ \sum_{j=1}^{n} \lambda_j x_{ij}^l \le \theta_o^l x_{io}^u, && i = 1,\ldots,m, \\
&\sum_{j=1}^{n} (\lambda_j - \mu_j) z_{pj}^l \ge 0, && p = 1,\ldots,q, \\
&\sum_{j=1}^{n} \mu_j y_{rj}^l \ge y_{ro}^l, && r = 1,\ldots,s, \\
&\theta_o^l \ge 0,\quad \lambda_j \ge 0,\ \mu_j \ge 0, && j = 1,\ldots,n.
\end{aligned} \quad \textbf{(14)}$$

$$\begin{aligned}
&\theta_o^{m^*} = \min\ \theta_o^m, \\
&\text{s.t.}\ \sum_{j=1}^{n} \lambda_j x_{ij}^m \le \theta_o^m x_{io}^m, && i = 1,\ldots,m, \\
&\sum_{j=1}^{n} (\lambda_j - \mu_j) z_{pj}^m \ge 0, && p = 1,\ldots,q, \\
&\sum_{j=1}^{n} \mu_j y_{rj}^m \ge y_{ro}^m, && r = 1,\ldots,s, \\
&\theta_o^m \ge 0,\ \lambda_j \ge 0,\ \mu_j \ge 0, && j = 1,\ldots,n.
\end{aligned} \quad \textbf{(15)}$$

$$\begin{aligned}
&\theta_o^{u^*} = \min\ \theta_o^u, \\
&\text{s.t.}\ \sum_{j=1}^{n} \lambda_j x_{ij}^u \le \theta_o^u x_{io}^l, && i = 1,\ldots,m, \\
&\sum_{j=1}^{n} (\lambda_j - \mu_j) z_{pj}^u \ge 0, && p = 1,\ldots,q, \\
&\sum_{j=1}^{n} \mu_j y_{rj}^u \ge y_{ro}^u, && r = 1,\ldots,s, \\
&\theta_o^u \ge 0,\ \lambda_j \ge 0,\ \mu_j \ge 0, && j = 1,\ldots,n.
\end{aligned} \quad \textbf{(16)}$$

Definition 4. ([8],[9]).

I. If

$$\tilde{\theta} = \left(\theta^l, \theta^m, \theta^u\right) = \tilde{1}.$$

that is,

$$\theta^l = \theta^m = \theta^u = 1.$$

then the DMU_o is strongly efficient.

II. If

$$\theta^l \leq 1,\ \theta^m = 1,\ \theta^u = 1.$$

then the $\mathrm{DMU_o}$ is efficient.

III. If

$$\theta^u = 1,\ \theta^l \leq 1,\ \theta^m \leq 1.$$

then the $\mathrm{DMU_o}$ is weakly efficient.

IV. If

$$\theta^l \leq 1,\ \theta^m \leq 1,\ \theta^u \leq 1.$$

Then the $\mathrm{DMU_o}$ is inefficient.

6.2 | DEA Cost Efficiency Model for a Two-Stage Process with Fuzzy Data

The envelopment form of the DEA cost efficiency model for a two-stage process proposed by Shiri Daryani et al. [15], [16] can be extended to fuzzy data and written as follows:

$$\begin{aligned}
& c^t\tilde{x}^* = \min\ \sum_{i=1}^{m} c_i \tilde{x}_i, \\
& \text{s.t.}\ \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \leq \tilde{x}_{io}, && i = 1,\ldots,m, \\
& \sum_{j=1}^{n} \lambda_j \tilde{z}_{pj} \geq \tilde{z}_{pj}, && p = 1,\ldots,q, \\
& \sum_{j=1}^{n} \mu_j \tilde{z}_{pj} \geq \tilde{z}_{pj}, && p = 1,\ldots,q, \\
& \sum_{j=1}^{n} \mu_j \tilde{y}_{rj} \geq \tilde{y}_{ro}, && r = 1,\ldots,s, \\
& \lambda_j \geq 0,\ \mu_j \geq 0, && j = 1,\ldots,n.
\end{aligned} \tag{17}$$

$$\mathrm{CE_o} = \frac{c^t\tilde{x}^*}{c^t\tilde{x}_o} = \frac{\sum_{i=1}^{m} c_i \tilde{x}_i^*}{\sum_{i=1}^{m} c_i \tilde{x}_{io}}. \tag{18}$$

Based on the triangular representation of the fuzzy data given in *Eq. (13)*, the following relations hold:

$$\begin{aligned}
& c^t\tilde{x}^* = \min\ \left(\sum_{i=1}^{m} c_i x_i^l, \sum_{i=1}^{m} c_i x_i^m, \sum_{i=1}^{m} c_i x_i^u \right), \\
& \text{s.t.} \left(\sum_{j=1}^{n} \lambda_j x_{ij}^l, \sum_{j=1}^{n} \lambda_j x_{ij}^m, \sum_{j=1}^{n} \lambda_j x_{ij}^u \right) \leq \left(x_i^l, x_i^m, x_i^u \right), && i = 1,\ldots,m, \\
& \left(\sum_{j=1}^{n} (\lambda_j - \mu_j) z_{kj}^l, \sum_{j=1}^{n} (\lambda_j - \mu_j) z_{kj}^m, \sum_{j=1}^{n} (\lambda_j - \mu_j) z_{kj}^u \right) \geq \left(0^l, 0^m, 0^u \right), && k = 1,\ldots,h, \\
& \left(\sum_{j=1}^{n} \mu_j y_{rj}^l, \sum_{j=1}^{n} \mu_j y_{rj}^m, \sum_{j=1}^{n} \mu_j y_{rj}^u \right) \geq \left(y_{ro}^l, y_{ro}^m, y_{ro}^u \right), && r = 1,\ldots,s, \\
& x_i^l,\ x_i^m - x_i^l \geq 0,\ x_i^u - x_i^m \geq 0, && i = 1,\ldots,m, \\
& \lambda_j \geq 0,\ \mu_j \geq 0, && j = 1,\ldots,n.
\end{aligned} \tag{19}$$

$$CE_o=\frac{c^t\tilde{x}^*}{c^t\tilde{x}_o}=\frac{\sum_{i=1}^m c_i\tilde{x}_i^*}{\sum_{i=1}^m c_i\tilde{x}_{io}}=\frac{\left(\sum_{i=1}^m c_ix_i^{l*},\ \sum_{i=1}^m c_ix_i^{m*},\ \sum_{i=1}^m c_ix_i^{u*}\right)}{\left(\sum_{i=1}^m c_ix_{io}^{l},\ \sum_{i=1}^m c_ix_{io}^{m},\ \sum_{i=1}^m c_ix_{io}^{u}\right)}=\left(\frac{\sum_{i=1}^m c_ix_i^{l*}}{\sum_{i=1}^m c_ix_{io}^{u}},\frac{\sum_{i=1}^m c_ix_i^{m*}}{\sum_{i=1}^m c_ix_{io}^{m}},\frac{\sum_{i=1}^m c_ix_i^{u*}}{\sum_{i=1}^m c_ix_{io}^{l}}\right)=\left(CE_o^l\ ,CE_o^m\ ,CE_o^u\right). \quad (20)$$

Using the fuzzy arithmetic approach, the upper, middle, and lower bounds of the cost efficiency are obtained as follows.

6.3 | Input Estimation Using the Inverse DEA Cost Efficiency Model for a Two-Stage Process with Fuzzy Data

$$\begin{aligned}
& c^tX^{l^*}=\min\ \sum_{i=1}^m c_ix_i^l \\
& \text{s.t.}\ \sum_{j=1}^n \lambda_j x_{ij}^l \le x_i^l, && i=1,...,m,\\
& \sum_{j=1}^n\left(\lambda_j-\mu_j\right)z_{pj}^u \ge 0, && k=1,...,h,\\
& \sum_{j=1}^n \mu_j y_{rj}^u \ge y_{ro}^u, && r=1,...,s,\\
& x_i^l \ge 0, && i=1,...,m,\\
& \lambda_j,\ \mu_j \ge 0, && j=1,\dots,n.
\end{aligned} \quad (21)$$

$$CE_o^l=\frac{\sum_{i=1}^m c_ix_i^{l*}}{\sum_{i=1}^m c_ix_{io}^u}. \quad (22)$$

$$\begin{aligned}
& c^tX^{m*}=\min\ \sum_{i=1}^m c_ix_i^m,\\
& \text{s.t.}\ \sum_{j=1}^n \lambda_j x_{ij}^m \le x_i^m, && i=1,\dots,m,\\
& \sum_{j=1}^n\left(\lambda_j-\mu_j\right)z_{pj}^m \ge 0^m, && k=1,\dots,h,\\
& \sum_{j=1}^n \mu_j y_{rj}^m \ge y_{ro}^m, && r=1,\dots,s,\\
& x_i^m \ge 0, && i=1,\dots,m,\\
& \lambda_j,\mu_j \ge 0, && j=1,\dots,n.
\end{aligned} \quad (23)$$

$$CE_o^m=\frac{\sum_{i=1}^m c_ix_i^{m*}}{\sum_{i=1}^m c_ix_{io}^m}. \quad (24)$$

$$\begin{aligned}
& c^tX^{u*}=\min\ \sum_{i=1}^m c_ix_i^u,\\
& \text{s.t.}\ \sum_{j=1}^n \lambda_j x_{ij}^u \le x_i^u, && i=1,\dots,m,\\
& \sum_{j=1}^n\left(\lambda_j-\mu_j\right)z_{pj}^l \ge 0, && k=1,\dots,h,\\
& \sum_{j=1}^n \mu_j y_{rj}^l \ge y_{ro}^l, && r=1,\dots,s,\\
& x_i^u \ge 0, && i=1,\dots,m,\\
& \lambda_j,\ \mu_j \ge 0, && j=1,\dots,n.
\end{aligned} \quad (25)$$

$$CE_o^u=\frac{\sum_{i=1}^m c_ix_i^{u*}}{\sum_{i=1}^m c_ix_{io}^l}. \quad (26)$$

Assume that the outputs of the $\mathrm{DMU_o}$ under evaluation, $\mathrm{DMU_o}$, increase from $\tilde{y}_{ro}$ to

$\tilde{\beta}_{ro} = (\tilde{y}_{ro} + \Delta\tilde{y}_{ro}),\ r = 1, \dots, s.$

To estimate the required changes in the input vector, namely,

$\tilde{\alpha} = (\tilde{x}_i + \Delta\tilde{x}_i),\ i = 1, \dots, m.$

While preserving the overall process efficiency and cost efficiency, the models presented below must be solved.

First, the cost efficiency after applying the changes to the outputs is computed using *Model (27).*

$$
\begin{aligned}
& c^t\overline{X} = \min \sum_{i=1}^{m} c_i\tilde{x}_i, \\
& \text{s.t. } \sum_{j=1}^{n} \lambda_j\tilde{x}_{ij} \le \tilde{x}_{io}, && i = 1,\dots,m, \\
& \sum_{j=1}^{n} \lambda_j\tilde{z}_{pj} \ge \tilde{z}_{pj}, && p = 1,\dots,q, \\
& \sum_{j=1}^{n} \mu_j\tilde{z}_{pj} \ge \tilde{z}_{pj}, && p = 1,\dots,q, \\
& \sum_{j=1}^{n} \mu_j\tilde{y}_{rj} \ge \left(\tilde{y}_{ro} + \Delta y_{ro}\right) = \tilde{\beta}_{ro}, && r = 1,\dots,s, \\
& \lambda_j \ge 0,\ \mu_j \ge 0, && j = 1,\dots,n.
\end{aligned}
\tag{27}
$$

The new input levels can then be estimated by solving the following model.

$$
\begin{aligned}
& \min\ \left(\tilde{\alpha}_{1o},\dots,\tilde{\alpha}_{mo}\right), \\
& \text{s.t. } \sum_{j=1}^{n} \lambda_j\tilde{x}_{ij} \le \tilde{\alpha}_{io}, && i = 1,\dots,m, \\
& \sum_{j=1}^{n} \lambda_j\tilde{z}_{pj} \ge \tilde{z}_{pj}, && p = 1,\dots,q, \\
& \sum_{j=1}^{n} \mu_j\tilde{z}_{pj} \ge \tilde{z}_{pj}, && p = 1,\dots,q, \\
& \sum_{j=1}^{n} \mu_j\tilde{y}_{rj} \ge \tilde{y}_{ro}, && r = 1,\dots,s, \\
& c^t\tilde{\alpha} = \frac{C^t\overline{x}}{CE}, \\
& \lambda_j \ge 0,\ \mu_j \ge 0\ , && j = 1,\dots,n.
\end{aligned}
\tag{28}
$$

Based on the triangular representation of the fuzzy data in *Eq. (13)*, the following relations hold:

$$
\begin{aligned}
& \min\ \left([\alpha_1^l,\alpha_1^m,\alpha_1^u],\dots,[\alpha_m^l,\alpha_m^m,\alpha_m^u]\right), \\
& \text{s.t. } \left(\sum_{j=1}^{n} \lambda_j x_{ij}^l, j\sum_{j=1}^{n} \lambda_j x_{ij}^m, \sum_{j=1}^{n} \lambda_j x_{ij}^u\right) \le \left(\theta_{input}^{*l},\theta_{input}^{*m},\theta_{input}^{*u}\right)\left(\alpha_{io}^l,\alpha_{io}^m,\alpha_{io}^u\right), && i = 1,\dots,m, \\
& \left(\sum_{j=1}^{n} \lambda_j z_{pj}^l, j\sum_{j=1}^{n} \lambda_j z_{pj}^m, \sum_{j=1}^{n} \lambda_j z_{pj}^u\right) \le \left(z_{po}^l, z_{po}^m, z_{po}^u\right), && p = 1,\dots,q, \\
& \left(\sum_{j=1}^{n} \mu_j y_{rj}^l, \sum_{j=1}^{n} \mu_j y_{rj}^m, \sum_{j=1}^{n} \mu_j y_{rj}^u\right) \ge \left(y_{rj}^l, y_{rj}^m, y_{rj}^u\right), && p = 1,\dots,q, \\
& \left(\sum_{i=1}^{m} c_i\alpha_{io}^l, \sum_{i=1}^{m} c_i\alpha_{io}^m, \sum_{i=1}^{m} c_i\alpha_{io}^u\right) = \frac{\left(\sum_{i=1}^{m} c_i\overline{x}_{io}^l\ ,\ \sum_{i=1}^{m} c_i\overline{x}_{io}^m\ ,\ \sum_{i=1}^{m} c_i\overline{x}_{io}^u\right)}{\left(CE^l, CE^m, CE^u\right)}\ , && i = 1,\dots,m, \\
& \lambda_j \ge 0,\ \mu_j \ge 0, && j = 1,\dots,n.
\end{aligned}
\tag{29}
$$

Using the fuzzy arithmetic approach, the optimal values corresponding to the upper, middle, and lower bounds can be determined by solving the following *Model (30).*

$$
\begin{aligned}
&\min \left(\alpha_1^l, \alpha_2^l, \ldots, \alpha_m^l\right), \\
&\text{s.t. } \sum\nolimits_{j=1}^{n} \lambda_j x_{ij}^u \leq \theta_{input}^u \alpha_{io}^l, && i=1,\ldots,m, \\
&\sum\nolimits_{j=1}^{n} \lambda_j z_{pj}^u \geq z_{po}^u, && p=1,\ldots,q, \\
&\sum\nolimits_{j=1}^{n} \mu_j z_{pj}^u \leq z_{po}^u, && p=1,\ldots,q, \\
&\sum\nolimits_{j=1}^{n} \mu_j y_{rj}^u \geq \beta_{ro}^u, && r=1,\ldots,s, \\
&CE^{u^*} = \frac{\sum_{i=1}^{m} c_i \overline{x}_i^{u^*}}{\sum_{i=1}^{m} c_i \alpha_i^l}, \\
&\lambda_j \geq 0,\ \mu_j \geq 0, && j=1,\ldots,n.
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
&\min \left(\alpha_{1o}^m, \alpha_{2o}^m, \ldots, \alpha_{mo}^m\right), \\
&\text{s.t. } \sum\nolimits_{j=1}^{n} \lambda_j x_{ij}^m \leq \theta_{input}^m \alpha_{io}^m, && i=1,\ldots,m, \\
&\sum\nolimits_{j=1}^{n} \lambda_j z_{pj}^m \geq z_{po}^m, \\
&\sum\nolimits_{j=1}^{n} \mu_j z_{pj}^m \leq z_{po}^m, \\
&\sum\nolimits_{j=1}^{n} \mu_j y_{rj}^m \geq \beta_{ro}^m, \\
&CE^{m^*} = \frac{\sum_{i=1}^{m} c_i \overline{x}_i^m}{\sum_{i=1}^{m} c_i \alpha_i^m}, && i=1,\ldots,m, \\
&\lambda_j \geq 0,\ \mu_j \geq 0, && j=1,\ldots,n.
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
&\min \left(\alpha_{1o}^u, \alpha_{2o}^u, \ldots, \alpha_{mo}^u\right), \\
&\text{s.t. } \sum\nolimits_{j=1}^{n} \lambda_j x_{ij}^l \leq \theta_{input}^l \alpha_{io}^u, && i=1,\ldots,m, \\
&\sum\nolimits_{j=1}^{n} \lambda_j z_{pj}^l \geq z_{po}^l, && p=1,\ldots,q, \\
&\sum\nolimits_{j=1}^{n} \mu_j z_{pj}^l \leq z_{po}^l, && p=1,\ldots,q, \\
&\sum\nolimits_{j=1}^{n} \mu_j y_{rj}^l \geq \beta_{ro}^l, && r=1,\ldots,s, \\
&CE^{l^*} = \frac{\sum_{i=1}^{m} c_i \overline{x}_i^{l*}}{\sum_{i=1}^{m} c_i \alpha_i^u}, && i=1,\ldots,m, \\
&\lambda_j \geq 0,\ \mu_j \geq 0, && j=1,\ldots,n.
\end{aligned}
\tag{32}
$$

Various approaches can be employed to solve the above multi-objective optimization model. As indicated in *Eq. (11)*, the weighted-sum method is employed in this study.

7 | Proposed Algorithm

Step 1. For the DMU_o under evaluation DMU_o, compute the TE using *Models (14)-(16)* based on the fuzzy arithmetic approach, thereby obtaining the lower, middle, and upper bounds of the efficiency score.

Step 2. Assume that $c_i \in \mathbb{R}(i,=,1 \ldots m)$ denotes the unit cost (weight) associated with inpu $x_i(i,=,1 \ldots m)$t. Then, solve *Models (21), (23),* and *(25),* and compute the cost efficiency scores for the lower, middle, and upper bounds using *Eqs. (22), (24)*, and *(26).*

Step 3. Assume that the outputs of the DMU_o under evaluation, DMU_o, change from $\tilde{y}_{ro}$ to:

$\tilde{\beta}_{ro} = (\tilde{y}_{ro} + \Delta\tilde{y}_{ro}),\ r = 1, \dots, s.$

Then, solve *Model (27)*.

Step 4. Solve *Models (30)* and *(32)* to estimate the new input vector:

$\tilde{\alpha} = (\tilde{x}_i + \Delta\tilde{x}_i),\ i = 1, \dots, m.$

For the lower, middle, and upper bounds. In this way, the overall process efficiency under the relational approach and the overall cost efficiency remain unchanged, while the output levels are allowed to vary.

8 | Numerical Example

The oil and petrochemical industry is one of the most influential sectors in the economies of both Iran and the world. Through a series of refining and processing operations performed on hydrocarbons in refineries, crude oil and natural gas are converted into primary, intermediate, and final products, which are widely used to meet essential demands in various sectors, including healthcare, food production, pharmaceuticals, and many other industries.

Stud bolts play a critical role in the petrochemical industry. These fastening components are essential for ensuring the safety, integrity, and reliability of mechanical joints during the construction, maintenance, and commissioning of oil and gas refineries. They are widely used to connect pipes, flanges, and storage tanks in oil and gas processing units for the transportation of fluids, gases, and chemical substances. Consequently, they must possess excellent resistance to high pressures, elevated temperatures, and corrosive chemical environments. It should be emphasized that even the slightest deficiency in the manufacturing, processing, or installation stages may result in severe financial losses and catastrophic safety hazards. Therefore, the manufacturing workshop carefully procures raw materials in accordance with international standards and utilizes advanced manufacturing technologies to produce industrial components. The finished products are subsequently subjected to rigorous testing by specialists in accredited laboratories before being packaged securely and delivered to refinery construction and commissioning sites, particularly those located in the southern regions of Iran, where qualified engineers and technicians install them.

In this study, the manufacturing workshop producing stud bolts and the application of its products in the installation of part of a refinery unit are selected as the case study.

Subprocess 1

I. Input 1: MO40 steel bar, size 16 (number of bars).

II. Input 2: MO40 steel bar, size 20 (number of bars).

III. Input 3: Labor (number of workers).

IV. Output 1 (intermediate measure): Stud bolts, Material: ASTM A193.

V. Output 2 (intermediate measure): Stud bolts, Material: ASTM A320.

Input costs

I. Purchase cost of each MO40 steel bar (size 16): 1,400,000 Iranian Tomans.

II. Purchase cost of each MO40 steel bar (size 20): 1,600,000 Iranian Tomans.

III. Average wage of each worker in this project: 7,000,000 Iranian Tomans.

Subprocess 2

I. Input 1: (intermediate measure): Stud bolts, material: ASTM A193.

II. Input 2: (intermediate measure): Stud bolts, material: ASTM A320.

Final output

Installation and connection of the fluid transmission pipeline (measured in meters), i.e., the total length of the connected pipeline.

Table 1. Dataset for the first subprocess.

No.	Input 1 Mo40 Steel Bar Size 16 (Number of Bars)	Input 2 Mo40 Steel Bar Size 20 (Number of Bars)	Input 3 Labor (Number of Workers)	Output 1 Intermediate Measure Stud Bolt (Astm A193)	Output 2 Intermediate Measure Stud Bolt (Astm A320)
1	(42, 45, 46)	(78, 80, 91)	(8, 9, 13)	(2100,2500,2800)	(4520,4700,5100)
2	(47, 50, 58)	(60, 77, 80)	(9, 10, 14)	(2800,3000,3500)	(3750,4010,4600)
3	(49, 52, 55)	(63, 68, 75)	(8, 10, 15)	(2720,3010,3050)	(3500,3900,4100)
4	(46, 48, 53)	(71, 75, 79)	(8, 10, 12)	(2640,3000,3010)	(3900,4100,4420)
5	(52, 55, 59)	(78, 80, 85)	(9, 12, 15)	(2920,3100,3500)	(3980,4600,4710)
6	(30, 40, 47)	(80, 82, 91)	(8, 11, 13)	(2100,2410,2600)	(4100,4780,4900)
7	(38, 45, 46)	(70, 81, 83)	(4, 5, 6)	(2910,3100,3540)	(4910,5100,5800)
8	(50, 52, 65)	(81, 82, 90)	(13, 14, 15)	(2120,2800,2920)	(2800,3000,3100)
9	(37, 39, 41)	(65, 70, 73)	(3, 4, 5)	(2340,2460,2600)	(3840,4200,4500)
10	(35, 40, 50)	(77, 79, 90)	(9, 10, 15)	(2090,2500,2900)	(4600,4720,5150)
11	(37, 41, 49)	(82, 83, 91)	(9, 11, 15)	(2100,2200,2800)	(4610,4630,5100)
12	(49, 53, 55)	(62, 69, 75)	(8, 9, 11)	(2710,3020,3100)	(3600,3950,4150)
13	(30, 32, 33)	(70, 71, 72)	(5, 6, 7)	(2600,2610,2700)	(4750,4800,4810)
14	(46, 51, 54)	(63, 70, 75)	(9, 10, 12)	(2712,3120,3200)	(3600,3960,4250)
15	(53, 54, 59)	(78, 81, 84)	(10, 13, 15)	(2930,3200,3500)	(3990,4700,4810)
16	(31, 41, 48)	(80, 83, 90)	(9, 10, 14)	(2100,2500,2620)	(4120,4800,4900)
17	(48, 51, 58)	(60, 72, 82)	(8, 11, 13)	(2900,3010,3510)	(3720,4012,4610)
18	(37, 40, 42)	(63, 70, 72)	(4, 5, 6)	(2320,2460,2610)	(3800,4320,4350)
19	(46, 53, 59)	(61, 80, 83)	(9, 11, 13)	(2810,3015,3525)	(3740,4000,4615)
20	(38, 39, 40)	(65, 70, 73)	(3, 5, 6)	(2340,2510,2598)	(3820,4310,4520)

Table 2. Dataset for the second subprocess.

No.	Input 1 (Intermediate Measure) Stud Bolt (Astm A193) (Number of Pieces)	Input 2 (Intermediate Measure) Stud Bolt (Astm A320) (Number of Pieces)	Final Output Installed Pipeline Connections (Meters)
1	(2100,2500,2800)	(4520,4700,5100)	(410,430,470)
2	(2800,3000,3500)	(3750,4010,4600)	(480,483,520)
3	(2720,3010,3050)	(3500,3900,4100)	(470,510,530)
4	(3012640,3000,0)	(3900,4100,4420)	(380,400,450)
5	(2920,3100,3500)	(3980,4600,4710)	(480,520,540)
6	(2100,2410,2600)	(4100,4780,4900)	(310,320,350)
7	(2910,3100,3540)	(4910,5100,5800)	(580,600,640)
8	(2120,2800,2920)	(2800,3000,3100)	(350,370,380)
9	(2340,2460,2600)	(3840,4200,4500)	(600,610,620)
10	(2100,2500,2900)	(4600,4720,5150)	(400,420,460)
11	(2100,2200,2800)	(4610,4630,5100)	(300,310,340)
12	(2710,3020,3100)	(3600,3950,4150)	(471,500,538)
13	(2310,2560,2700)	(3810,4310,4600)	(700,730,780)
14	(2712,3120,3200)	(3600,3960,4250)	(500,510,545)
15	(2930,3200,3500)	(3990,4700,4810)	(470,530,541)
16	(2100,2500,2620)	(4120,4800,4900)	(310,330,360)
17	(2900,3010,3510)	(3720,4012,4610)	(480,489,520)
18	(2320,2460,2610)	(3800,4320,4350)	(600,612,625)
19	(2810,3015,3525)	(3740,4000,4615)	(475,503,527)
20	(2340,2510,2598)	(3820,4310,4520)	(610,620,631)

Table 3. TE table.

DMU	θ^{*l}_{input}	θ^{*m}_{input}	θ^{*u}_{input}
1	0.31	0.39	0.40
2	0.40	0.45	0.57
3	0.42	0.54	0.55
4	0.33	0.39	0.42
5	0.38	0.45	0.47
6	0.24	0.28	0.37
7	0.48	0.57	0.76
8	0.26	0.30	0.33
9	0.56	0.71	0.98
10	0.30	0.39	0.42
11	0.22	0.22	0.27
12	0.42	0.52	0.57
13	0.71	0.75	0.86
14	0.45	0.53	0.57
15	0.38	0.45	0.48
16	0.24	0.29	0.37
17	0.39	0.49	0.57
18	0.56	0.66	0.75
19	0.39	0.45	0.57
20	0.57	0.67	1

Table 4. Cost efficiency table.

DMU	CE^{*l}	CE^{*m}	CE^{*u}
1	0.32	0.46	0.51
2	0.35	0.50	0.64
3	0.36	0.56	0.63
4	0.32	0.43	0.49
5	0.34	0.49	0.55
6	0.24	0.33	0.41
7	0.55	0.72	0.90
8	0.23	0.34	0.36
9	0.61	0.86	1
10	0.30	0.46	0.51
11	0.22	0.32	0.37
12	0.40	0.55	0.63
13	0.76	1	1
14	0.40	0.55	0.66
15	0.34	0.49	0.52
16	0.24	0.35	0.40
17	0.35	0.51	0.66
18	0.59	0.83	1
19	0.35	0.49	0.63
20	0.60	0.84	1

In *Table 3*, the TE values of all DMUs (workshops) are calculated at the lower, middle, and upper bounds using the proposed models. As shown in *Table 3*, the upper-bound TE value of DMU_o 20 is equal to one. According to *Definition 1*, this indicates that this unit is weakly efficient. The other DMUs are inefficient Furthermore, suppose that the input cost vector is given by $(c_1, c_2, c_3) = (7, 1.6, 1.4)$. The cost efficiency scores were calculated using *Models (21)–(26)* for the lower, middle, and upper bounds. DMUs 9 and 20 obtained a cost efficiency score of one at the upper bound and are therefore cost-efficient at the upper bound. DMU_o 13 also obtained a cost efficiency score of one at both the middle and upper bounds, which may indicate that this unit is cost-efficient.

8.1 | Suppose the Manager Decides to Change the Outputs of DMUs (Workshops)

Managers or engineers who are interested in increasing the output levels based on their desired preferences are considered. The question is: by how much should the input values be changed so that the overall TE and cost efficiency remain unchanged, while the output levels change to the desired levels?

The TE values of all DMUs (workshops) in the three lower, middle, and upper bounds have been kept constant according to *Table 5*. The output levels of the workshops based on the preferences of the managers and engineers are also presented in *Table 5*.

Table 5. Changes in the outputs of manager-selected DMUs.

DMU	$\left(\theta^{l^*},\theta^{m^*},\theta^{u}\right)$	$\left(CE^{l^*},CE^{m^8},CE^{u^*}\right)$	$\left(\beta^{l},\beta^{m},\beta^{u}\right)$
DMU7	(0.48,0.57,0.76)	(0.55,0.72,0.90)	(590,720,900)
DMU9	(0.56,0.71,0.98)	(0.61,0.86,1)	(620,730,900)
DMU13	(0.71,0.75,0.86)	(0,76,1,1)	(710,850,950)
DMU19	(0.39,0.45,0.57)	(0.35,0.49,0.63)	(480,504,800)
DMU20	(0.57,0.67,1)	(0.60,0.84,1)	(612,800,900)

Table 6. Input estimation using the proposed model.

DMU	$\left(\alpha_1^{l},\alpha_2^{l},,\alpha_3^{l}\right)$	$\left(\alpha_1^{m},\alpha_2^{m},,\alpha_3^{m}\right)$	$\left(\alpha_1^{u},\alpha_2^{u},,\alpha_3^{u}\right)$
DMU7	(75,41,4)	(45,97,8)	(62,114,10)
DMU9	(65,37,5)	(39,80,7)	(55,103,9)
DMU13	(40,73,5)	(41,82,6)	(50,92,7)
DMU19	(46,77,9)	(53,80,11)	(63,119,19)
DMU20	(38,65,5)	(41,92,8)	(54,99,10)

The results presented in *Table 6* show that if the output levels, for example, for DMU_0 7 (workshop 7), are increased to (590,720,900), the required input values for the lower, middle, and upper bounds are, respectively, (4,75,41), (45,97,8), and (62,114,10) for the first, second, and third inputs. These findings enable managers to make more informed decisions regarding the provision of raw materials and labor, particularly in critical industries such as the petrochemical industry. Furthermore, this approach provides a basis for developing more realistic plans to achieve self-sufficiency in production. Using this information, the stages of the production process can be organized in advance of implementation; consequently, quality can be improved, waste can be reduced, and more desirable outcomes can be achieved.

9 | Conclusion

Since, in reality, public and private organizations such as hospitals, schools, banks, and others often consist of two-stage or multi-stage processes, network DEA models should be employed to evaluate their efficiency. Classical DEA models treat the system as a black box and do not seek to identify the sources of inefficiency within individual subprocesses. In recent decades, new network models have been developed that incorporate intermediate measures. On the other hand, in empirical studies, data are not always precise and deterministic; they may be ambiguous (fuzzy) and represented by triangular fuzzy numbers with lower, middle, and upper bounds. Consequently, network models with fuzzy data have also been developed, and various approaches have been proposed for solving them. In this study, a NDEA model for TE with triangular fuzzy data is proposed and solved using a fuzzy arithmetic approach. In this method, three linear programming problems are readily solved to obtain the bounds of the TE of the evaluated unit. Accordingly, a cost-efficiency model is also proposed, and the best values of cost efficiency are calculated at the three bounds. Subsequently, an IDEA model with triangular fuzzy representations of inputs and outputs is introduced, such that the overall technical and cost efficiencies are preserved. In contrast, the output vector is perturbed, and the required changes in inputs are estimated using the proposed model. This model provides a powerful tool that enables managers not only to evaluate past performance but also to make informed and well-grounded decisions with regard to future planning.

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Data Availability

Sufficient data are available upon request.

Conflict of Interest

No potential conflict of interest was reported by the authors.

Consent for Publication

All authors have provided their consent for the publication of this manuscript.

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