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Shared Investment for Network Flow under Fuzzy Membership Scenarios

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Abstract

Capacity improvement is often planned from one network assessment even though edge suitability varies across operating conditions. We study a common investment vector that supports a required terminal flow in every member of a finite collection of fuzzy membership scenarios. Routing can change with the scenario, while the installed capacity increments are shared. The model is a linear program with explicit investment caps. We prove equivalence with scenario dependent cut inequalities, derive a dual cost bound and characterize the largest feasible service target. The optimal investment cost is a nondecreasing piecewise linear convex function of that target. A six vertex example shows that a design requiring no investment under the nominal assessment can miss the requested service in other scenarios; a shared investment of 10 normalized units meets all three requirements. Forty generated perturbations compare nominal and shared designs using identical cost data. The formulation combines a clear membership to capacity conversion with independently checkable flow and cut conditions. The worked optimum also admits a direct lower bound from three disjoint sets of cut arcs, so its cost can be verified by hand. The analysis distinguishes common installation from adaptive routing and explains when a componentwise smallest scenario makes nominal design sufficient. Scenario coverage is conditional on the chosen assessments.

Keywords: Fuzzy network, Capacity expansion, Network flow, Scenario optimization, Linear programming, Cut inequality.

1| Introduction

Communication and distribution networks may operate under several conditions: ordinary demand, partial degradation and restricted access. An assessment of edge suitability can change between those conditions. An investment decision is usually made before the operating condition is observed, whereas flow can be rerouted afterwards. A planning model should therefore distinguish a common installed improvement from scenario dependent operating decisions.

The distinction matters even in a small network. A nominal flow of nine units may be feasible without investment, while two alternative capacity assessments support only seven. Increasing every edge equally is possible but can waste resources away from the limiting cuts. The relevant question is the least common investment that removes all specified service deficits.

Maximum flow and minimum cut duality are classical [1]. Minimum cost network expansion is also established [2], and optimization with uncertain coefficients has a broad literature [3]. Here we formulate a capped

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membership scenario model, state its exact flow and cut descriptions, and give a dual lower bound whose components can be checked separately. The formulation is a finite scenario linear program whose feasibility is determined by maximum flow duality.

1.1| Related Literature and Modelling Distinctions

Bellman and Zadeh [5] formulate decisions with fuzzy goals and constraints through membership functions. The present model uses a narrower interpretation: a membership is converted explicitly into usable capacity, after which each scenario imposes ordinary linear constraints. It neither optimizes a fuzzy goal membership nor treats a membership value as a probability. This distinction matters when translating an assessment of suitability into a physical service requirement.

Set-based protection against uncertain coefficients has a long history, including Soyster's set-inclusive formulation [6]. Bertsimas and Sim [7] study robust discrete optimization and network flows with a controllable degree of conservatism; their later price-of-robustness framework [3] develops that trade-off for linear optimization. These works motivate specifying the protected uncertainty explicitly. Here the protection set is the supplied finite collection of capacity vectors, not a budgeted uncertainty set, and no probability of out-of-sample feasibility is inferred from it.

The distinction between an installation fixed in advance and routing chosen after a scenario is known is an instance of adjustable robust optimization [8]. Atamtürk and Zhang [9] study two-stage network flow and design with uncertain demand and show why recourse and the structure of the uncertainty set affect tractability. Our uncertainty is instead in a finite list of base-capacity assessments, the terminal target is fixed, and all operating flows can be written explicitly. Thus the linear formulation here is a restricted, directly inspectable model rather than a general tractability result for two-stage robust network design.

Classical flow algorithms such as those of Edmonds and Karp [10] provide the algorithmic setting for checking a fixed investment scenario by scenario. Eliminating operating variables and adding violated feasibility conditions also has a connection to Benders' partitioning principle [11]. In this particular single-commodity setting those conditions are terminal-cut inequalities, separated by minimum-cut computations. The later cut-generation discussion describes this structural connection; it does not claim that the archived numerical study implemented a Benders solver.

2| Membership Scenarios and Capacity Interpretation

Let $G = (V, E)$ be a directed network with source s and sink t. Let $K_e > 0$ be a fixed reference capacity. Under scenario $r \in \{1, \dots, R\}$, edge membership $\mu_e^r \in [0, 1]$ determines the usable base capacity

$$b_e^r = K_e \mu_e^r. \tag{1}$$

This conversion is an explicit assumption: membership represents the fraction of a reference service capacity regarded as usable in the assessment. It is not a failure probability. A common installed increment x_e has unit cost $k_e > 0$ and cap $U_e \geq 0$. We require

$$0 \leq x_e \leq U_e, \qquad b_e^r + U_e \leq K_e \quad \text{for all } r. \tag{2}$$

Thus final membership is $(b_e^r + x_e)/K_e \leq 1$. The same physical increment is available in every scenario. A scenario describes a possible assessment, without an assigned probability.

Let B be the vertex–arc incidence matrix, with $+1$ at an arc's tail and -1 at its head. Put $d = e_s - e_t$. A required flow $q \geq 0$ is supported when, for every r, there exists f^r satisfying $Bf^r = qd$ and $0 \leq f^r \leq b^r + x$.

The shared design problem is

$$\begin{aligned} v(q) = \min_{x, f^1, \dots, f^R} \quad & k^T x \\ \text{subject to} \quad & Bf^r = qd, \quad 0 \leq f^r \leq b^r + x \quad (r = 1, \dots, R), \\ & 0 \leq x \leq U. \end{aligned} \tag{3}$$

The number of variables is $(R+1)m$. The scenario flows are operational recourse: they need not be identical. Requiring a common flow would define a different and generally more restrictive problem.

3| Cut Characterization and Feasibility

For $S \subset V$ with $s \in S$ and $t \notin S$, write $\delta^+(S)$ for arcs leaving S.

Theorem 1 (Scenario cut equivalence). *An investment $x \in [0, U]$ supports q in every scenario if and only if*

$$\sum_{e \in \delta^+(S)} (b_e^r + x_e) \geq q \quad \text{for every } r \text{ and every terminal cut } S. \tag{4}$$

Proof: If a scenario flow exists, summing its conservation equations over S shows that its net outgoing flow is q. Outgoing flow is bounded by the outgoing capacity, so (4) is necessary. Conversely, if all terminal cuts have capacity at least q, the maximum flow–minimum cut theorem [1] gives a flow of value at least q in that scenario. Removing cycles and reducing path flows, or scaling the flow, produces value exactly q. Apply the argument independently to every scenario. □

Corollary 1 (Largest feasible target). *The largest feasible service target is*

$$q_{\max} = \min_r \min_{S: s \in S, t \notin S} \sum_{e \in \delta^+(S)} (b_e^r + U_e). \tag{5}$$

Proof: No investment can exceed the capacities obtained from $x = U$, giving necessity. If $q \leq q_{\max}$, the investment $x = U$ meets all cut inequalities and is feasible by Theorem 1. □

This feasibility test should precede optimization. Otherwise a solver failure can be confused with an impossible target. It also identifies which scenario and cut limit further service even when every permitted upgrade is installed.

4| Duality and Target Sensitivity

For each scenario choose vertex potentials $\pi^r \in \mathbb{R}^n$ and nonnegative arc multipliers $\alpha^r \geq 0$. Let $\beta \geq 0$ price the investment caps. The dual is

$$\begin{aligned} \max_{\pi, \alpha, \beta} \quad & q \sum_r d^T \pi^r - \sum_r (b^r)^T \alpha^r - U^T \beta \\ \text{subject to} \quad & B^T \pi^r \leq \alpha^r \quad (r = 1, \ldots, R), \\ & \sum_r \alpha^r - \beta \leq k, \qquad \alpha^r, \beta \geq 0. \end{aligned} \tag{6}$$

Proposition 1 (Cost lower bound). *Every feasible vector in* (6) *gives a lower bound on the cost of every feasible design. If the primal is feasible, its minimum equals the maximum of* (6).

Proof: For each scenario,

$$q d^T \pi^r = (f^r)^T B^T \pi^r \leq (f^r)^T \alpha^r \leq (b^r + x)^T \alpha^r.$$

Summing and subtracting base capacity terms gives a bound by $x^T \sum_r \alpha^r \leq k^T x + \beta^T x \leq k^T x + \beta^T U$. Rearrangement proves weak duality. Strong duality follows from finite dimensional linear programming [4]: the primal is feasible, its investment variables are bounded and its objective is finite. Alternatively, the displayed dual follows directly by minimizing the Lagrangian over nonnegative flow and investment variables. □

The identity $\sum_r \alpha_e^r \leq k_e + \beta_e$ shows why optimizing scenarios independently and adding their investments can overpay. The same edge increment can support several operating conditions, and its combined scarcity value must be compared with one installation cost.

Theorem 2 (Shape of the target cost). *On $[0, q_{\max}]$, $v(q)$ is nondecreasing, convex and piecewise linear. If (π^r, α^r, β) is dual optimal at q, then*

$$g = \sum_r d^T \pi^r \quad \text{satisfies} \quad v(q') \geq v(q) + g(q' - q) \tag{7}$$

for every feasible q'.

Proof: Any feasible higher value flow can be scaled down, proving monotonicity of minimum cost. A convex combination of two feasible designs and their scenario flows is feasible at the corresponding convex combination of targets; its cost is the same combination of costs. Hence v is convex. Parametric linear programming, or the finite set of dual basic solutions after fixing redundant potentials, makes the value

piecewise linear. Keeping a dual optimizer fixed at q gives an affine lower bound in q' and proves the subgradient inequality. □

If one scenario has no larger base capacity than another on every edge, satisfying the smaller scenario automatically satisfies the larger. Such dominated scenarios may be removed. Incomparable scenarios generally remain necessary. A single "average membership" scenario can erase the critical cut of each component scenario and has no all scenario service interpretation.

5| Solution Procedure

The design and verification procedure is as follows.

(1) Validate memberships, reference capacities, costs and investment caps.
(2) Compute $q_{\max}$ using one maximum flow problem per scenario at $x = U$.
(3) If the target is feasible, solve (3) with a linear programming method.
(4) For the returned investment, compute an independent maximum flow and minimum cut in every scenario.
(5) Recover a dual solution and report primal feasibility, dual feasibility and the objective gap.

For very many scenarios, a cut generation version begins with a subset of (4). The current investment is separated by a minimum cut in each scenario. Every violated cut is added and the investment problem is solved again. Since there are finitely many cuts, exact arithmetic termination is finite. This observation does not give a small worst case number of iterations.

6| Worked Three Scenario Network

The example is a constructed distribution network, not field data. The nine arcs and all three base capacities appear in Table 1. Set $K_e = \max_r b_e^r + U_e$ to obtain the corresponding fuzzy memberships. The target is $q = 9$ units.

Table 1. Complete network inputs and the shared optimum at target nine.

Arc	Base 1	Base 2	Base 3	Cap	Cost	Investment
1→2	4	2	4	3	3	0.000
1→3	5	5	3	3	2	2.000
2→3	2	2	1	2	1	0.000
2→4	4	2	4	3	2	0.000
3→4	2	2	1	2	1	2.000
3→5	4	4	2	3	3	0.000
4→5	2	2	2	2	1	0.000
4→6	5	3	5	3	2	2.000
5→6	4	4	2	3	2	0.000

The computed optimum invests two units in arcs $1 \to 3$, $3 \to 4$ and $4 \to 6$. Their unit costs are 2, 1 and 2, respectively, so total cost is

$$2(2) + 1(2) + 2(2) = 10. \tag{8}$$

The value can also be certified without trusting the optimizer. In scenario 2, the source cut has base capacity seven, so $x_{12} + x_{13} \geq 2$; these increments cost at least four. In scenario 3, the cut leaving $\{1, 2, 3\}$ has base capacity seven, so $x_{24} + x_{34} + x_{35} \geq 2$; its increments cost at least two. The sink cut in scenario 3 also has base capacity seven, so $x_{46} + x_{56} \geq 2$; its increments cost at least four. The three sets of priced arcs are disjoint. Adding their lower bounds proves that every feasible common investment costs at least ten. The listed design attains that lower bound. The nominal assessment alone needs no investment. Table 2 shows the resulting maximum flows in all scenarios. Thus a nominally adequate design can fail the stated all scenario requirement.

The primal and dual objective values agree within numerical tolerance. Figure 2 shows the service cost curve. Its increasing slopes describe successive constraints becoming active; they do not describe a probability of service failure.

Table 2. Maximum terminal flow after each investment rule.

Scenario	No investment	Nominal design	Shared design
1	9.000	9.000	11.000
2	7.000	7.000	9.000
3	7.000	7.000	9.000

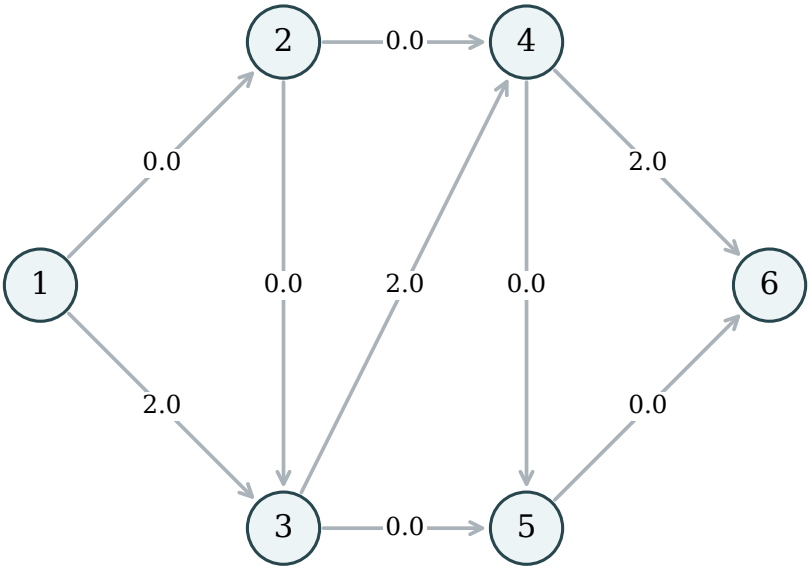


Figure 1. Directed illustrative network. Labels show installed increments at the shared optimum.

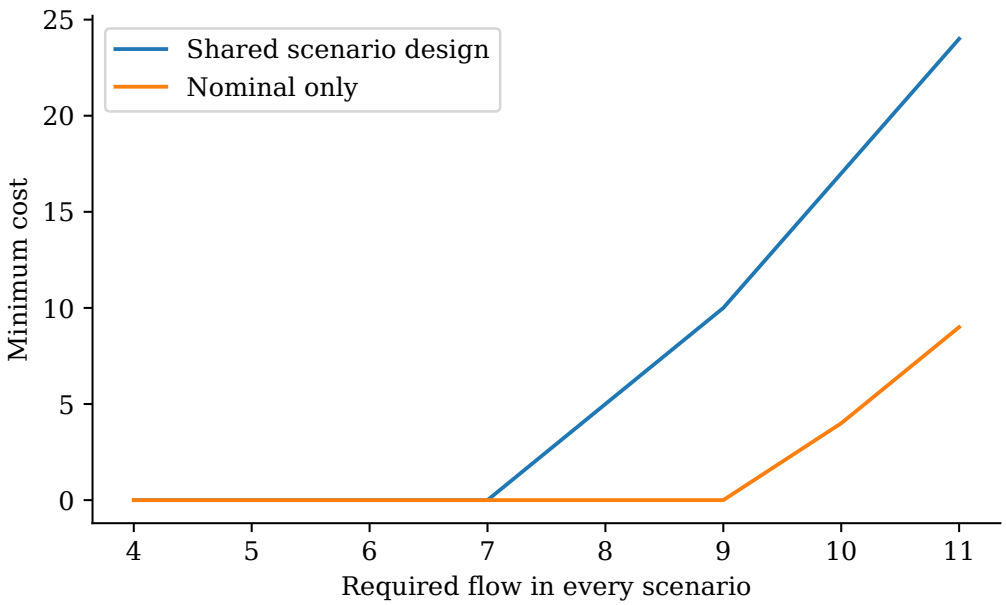


Figure 2. Minimum investment as the required flow increases.

7| Generated Perturbation Study

We keep the topology and caps fixed and create 40 capacity and price instances. Each base capacity is multiplied independently by a uniform number in $[0.85, 1.15]$. Each unit cost is multiplied by a uniform number in $[0.8, 1.2]$. Seeds are $2200, \ldots, 2239$. For each instance define q_0 as the smallest scenario maximum flow before investment and set

$$q = q_0 + 0.55(q_{\max} - q_0). \tag{9}$$

This keeps the target within the capacity model's feasible interval and gives a comparable degree of service expansion across instances. We solve both the nominal design using the first scenario and the common design using all three. We then evaluate both investments against every scenario. For each perturbed instance set $K_e = \max_r b_e^r + U_e$ and $\mu_e^r = b_e^r / K_e$. This gives memberships in $[0, 1]$ and ensures $b_e^r + U_e \le K_e$ for every scenario. The reference capacities are defined separately for each instance; the optimization uses the resulting capacity arrays directly.

Table 3. Forty generated instances. Infeasibility means that at least one specified scenario misses its instance target by more than 10^{-7}.

Design	Mean cost	Mean worst flow	Infeasible instances
Nominal	3.238	6.710	40
Shared	16.312	9.789	0

The largest absolute primal–dual cost discrepancy across the runs is below 2×10^{-14} in the recorded computation. The supplementary verification reconstructs every instance, enumerates all terminal cuts, and solves both the flow formulation and the cut formulation. The largest primal feasibility residual is below 10^{-15}, the largest dual feasibility residual is below 5×10^{-16}, and the largest cost gap is below 2×10^{-14}. These checks include the signs of the multipliers and all investment caps. The nominal rule is cheaper because it solves a weaker problem. Cost alone therefore cannot decide which rule is preferable without specifying the service obligation.

8| Discussion and Conclusion

8.1| A Condition under Which Nominal Design Suffices

If a nominal scenario is componentwise no larger than all other scenarios, every flow feasible in the nominal network remains feasible in the others for the same installed investment. In that special case the nominal optimum is a shared optimum. The example deliberately uses incomparable scenarios, where different cuts lose capacity. This explains why the observed extra cost is caused by a specific uncertainty structure, rather than by the mere presence of more than one scenario.

8.2| Scope of the Result

The formulation supplies a common investment, separate feasible operating flows and an explicit lower bound on total cost. It is appropriate when capacity increments are continuously divisible and retain the same installed benefit across the chosen scenarios. Discrete equipment sizes, fixed activation costs, uncertain investment effectiveness and congestion dependent capacities require extended models.

The finite scenario set must be justified by the application. Passing the listed scenarios does not establish performance under all possible future conditions. Conversely, multiplying several subjective memberships without a physical interpretation would obscure the capacity meaning. The explicit relation $b_e^r = K_e \mu_e^r$ makes the assumption inspectable and replaceable.

This study shows how fuzzy capacity assessments and classical network duality can support a transparent expansion decision. Further research may include scenario selection, integer equipment choices and multiple commodities, where the simple cut characterization no longer suffices in this form.

Data and Computation

The supplementary program reconstructs the perturbed base-capacity array in row-major scenario order before drawing the price multipliers. It verifies the flow and cut formulations, membership normalization

and dual feasibility. CSV files labelled `regenerated` contain the actual random seeds; the original file uses a zero-based instance index in the first column. All numerical inputs are synthetic and the complete worked network is listed. Linear programs are solved using SciPy's HiGHS interface. The analysis reports service feasibility and objective values, without claiming measured transport or communication performance.

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