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Minimum Cost Membership Upgrades for Fuzzy Path Connectivity

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Abstract

A fuzzy network may contain a route between two terminals while every available route includes an edge below a required membership level. We study the least investment needed to raise the strength of at least one route to a prescribed value. Each edge has an initial membership, an upper limit and a nondecreasing upgrade cost. The target problem is reduced exactly to an ordinary shortest path problem whose edge lengths are the costs of reaching the target. We derive a path optimality condition, a feasible budget search and an error bound for uncertain initial memberships. A counterexample shows that the optimal target cost can be nonconvex even when all individual upgrade costs are convex. A six vertex example and 500 reproducible tests compare the method with routes selected by hop count and initial bottleneck strength. Exhaustive enumeration agrees with the reduction in every test. The study provides an explicit investment interpretation of fuzzy path strength and identifies the conditions under which a single route design can be solved without a general nonlinear network optimizer. The reduction allows unequal prices and nonlinear costs without changing the route algorithm. Budget selection uses monotone feasibility tests, and the sensitivity result quantifies how a bounded error in edge assessment can affect the required expenditure.

Keywords: Fuzzy graph, Path connectivity, Network improvement, Shortest path, Sensitivity analysis.

1| Introduction

An edge membership can describe how adequately a communication link, a road segment or a service relation meets a stated operating requirement. A route is only as strong as its weakest edge when the minimum rule is used to combine these memberships. In a maintenance problem the decision is therefore not simply which route has the largest current strength. The decision is where to invest so that a required route strength is obtained at the lowest cost.

Consider two routes. One has strong edges that are expensive to improve. The other contains a weak edge that is inexpensive to repair. Ranking routes before accounting for repair costs can select the more expensive intervention. A model should place membership requirements and intervention costs in the same optimization problem. It should also give a proof that no other feasible route can be improved more cheaply.

Fuzzy sets and fuzzy graphs provide the membership language [1], [2]. Network improvement with edge dependent costs is an established optimization subject [3]. Our contribution is a focused formulation for one terminal pair with a minimum membership requirement, together with an explicit reduction, sensitivity statements and a reproducible comparison of three investment rules. The shortest path algorithm itself is classical [4]. The reduction is useful because it specifies precisely which nonlinear looking upgrade problems retain that simple structure.

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2| Model and Relation to Existing Methods

Let $G = (V, E)$ be a finite undirected graph, with $n = |V|$, $m = |E|$ and distinct terminals s, t. Vertices have membership one. Edge e has membership $\mu_e \in [0, 1]$ and an admissible upper limit $\bar{\mu}_e \in [\mu_e, 1]$. An upgrade x_e satisfies $0 \le x_e \le \bar{\mu}_e - \mu_e$. Its cost $c_e(x_e)$ is nonnegative and nondecreasing, with $c_e(0) = 0$.

For a simple route P, define

$$b(P, x) = \min_{e \in P}(\mu_e + x_e), \qquad b_{st}(x) = \max_{P: s \rightsquigarrow t} b(P, x). \tag{1}$$

An absent route has strength zero. For a target $\tau \in (0, 1]$, the investment problem is

$$C(\tau) = \min_{x} \left\{ \sum_{e \in E} c_e(x_e) : b_{st}(x) \ge \tau, \quad 0 \le x_e \le \bar{\mu}_e - \mu_e \right\}. \tag{2}$$

The membership is a degree of suitability, not a probability of independent edge survival. Consequently, the minimum operator in (1) is a modelling choice and does not estimate an end to end failure probability.

An ordinary shortest route minimizes fixed edge lengths. A widest route maximizes the initial minimum membership. Eq. (2) instead minimizes the cost of changing that membership. The three decisions coincide only in special cases. The present model also differs from network improvement problems that require all vertex pairs, a spanning tree or several disjoint routes to meet a condition: investments can then be shared across requirements, and the single route reduction below does not directly apply.

2.1| Related Literature

Strong-path structure and investment selection answer different questions. Bhutani and Rosenfeld characterize strong arcs through the strength of connectedness of their endpoints [5]. Mathew and Sunitha develop fuzzy node and arc connectivity through strength-reducing cuts [6]. These structural notions help describe the current network; the present objective additionally attaches an intervention cost to obtaining a specified terminal connection. It does not require every selected edge to be strong before investment.

A separate literature treats arc lengths themselves as fuzzy numbers. Okada and Soper use a fuzzy-number ordering and a multiple-labeling method to obtain nondominated paths [7]. Here memberships are real degrees of suitability, and upgrade costs are real-valued functions. Fixing the required membership therefore gives scalar additive edge costs rather than a fuzzy-number path-ranking problem. This distinction explains why an ordinary shortest path routine is appropriate under the stated assumptions.

Network upgrading also includes decisions with shared effects. Krumke and coauthors study node upgrades that change the weights of incident edges while imposing a spanning-tree requirement [8]. Such coupled interventions differ from the independent edge costs used here. More closely related to the threshold construction, Sepasian and Monabbati study bounded edge modifications for a min–max spanning tree and convert a fixed bottleneck target into a minimum-sum spanning-tree problem [9]. Their spanning requirement and the present single terminal-pair requirement lead to different combinatorial subproblems; the common principle is to price the edge modifications needed for a fixed target.

Changing the target also connects the model to parametric shortest paths. Karp and Orlin consider a common additive parameter on selected arcs [10], and Young, Tarjan and Orlin improve the corresponding parametric algorithm [11]. Our transformed costs may instead be nonlinear functions of the target, and upper membership limits can remove edges. We therefore use a shortest path solve at each tested target and establish the required monotonicity directly, rather than importing the running-time guarantees of those specialized parametric algorithms.

3| Exact Target Reduction

For each target, delete every edge with $\bar{\mu}_e < \tau$ and give the remaining edges the lengths

$$\ell_e(\tau) = c_e\big([\tau - \mu_e]_+\big). \tag{3}$$

All these lengths are nonnegative, even when costs are neither linear nor convex.

Theorem 1 (Target cost identity)**.** *If s and t are connected in the remaining graph, then*

$$C(\tau) = \min_{P: s \rightsquigarrow t} \sum_{e \in P} \ell_e(\tau). \tag{4}$$

For a minimizing route P^*, an optimal upgrade is $x_e = [\tau - \mu_e]_+$ on P^* and $x_e = 0$ elsewhere. If no such route exists, the target is infeasible.

Proof: Any feasible upgrade contains a route P with $\mu_e + x_e \geq \tau$ for every $e \in P$. Hence $\bar{\mu}_e \geq \tau$ and $x_e \geq [\tau - \mu_e]_+$ on this route. Monotonicity of each cost gives

$$\sum_{e \in E} c_e(x_e) \geq \sum_{e \in P} c_e(x_e) \geq \sum_{e \in P} \ell_e(\tau) \geq \min_{Q} \sum_{e \in Q} \ell_e(\tau).$$

Conversely, the stated upgrade on a minimizing route obeys every upper limit and attains route strength at least τ. Its total cost equals the last minimum. The same argument shows infeasibility when the deleted graph has no terminal route. □

Corollary 1 (An optimality condition)**.** *Suppose $d_s = 0$ and $d_v \leq d_u + \ell_{uv}(\tau)$ for both orientations of every retained edge. If a route P satisfies $\sum_{e \in P} \ell_e(\tau) = d_t$, its induced upgrade is optimal.*

Proof: Summing the distance inequalities along any terminal route gives $d_t \leq \sum_{e \in P} \ell_e(\tau)$. Equality for one feasible route proves optimality by Theorem 1. □

Thus the route, its upgrade amounts and a distance label at each vertex form a compact optimality check. Checking the edge inequalities takes $O(m)$ operations after the candidate is supplied. Dijkstra's algorithm computes the distance labels in $O((n + m) \log n)$ time with a binary heap. Cycles can be removed from any candidate walk without increasing cost, so simple routes are sufficient.

4| Target Sensitivity and Budget Selection

Proposition 1 (Monotonicity and local regularity)**.** *$C(\tau)$ is nondecreasing on its feasible target domain. Fix an interval $I \subseteq (0, 1]$ on which the admissible edge set is constant and contains an s–t route. If all edge costs are continuous, then C is finite and continuous on I. If each composed cost ℓ_e is L_e Lipschitz continuous on I, then, for every $\tau, \tau' \in I$,*

$$|C(\tau) - C(\tau')| \leq \max_{P} \sum_{e \in P} L_e \, |\tau - \tau'|. \tag{5}$$

Here the maximum ranges over the simple terminal routes in this fixed admissible graph.

Proof: Increasing the target shrinks the feasible upgrade set, proving monotonicity. The remaining claims follow from writing C as the minimum of finitely many continuous route costs. Select a minimizing route at τ' to obtain one direction of the Lipschitz inequality, and interchange the two targets for the other direction. □

An upper limit can cause a jump when a previously economical route becomes inadmissible. When all upper limits are one, the admissible edge set is unchanged for every $\tau \in (0, 1]$. Convex edge costs, however, do not imply convex optimal target cost. Consider two internally disjoint routes, each with one improvable edge and a second edge of membership one and zero upgrade cost. The improvable edges have $(\mu, a) = (0.2, 10)$ and $(0.6, 30)$ with linear cost ax. Then

$$C(\tau) = \min\{10\,[\tau - 0.2]_+ \,, 30\,[\tau - 0.6]_+\}. \tag{6}$$

At $\tau = 0.8$ the active slope decreases from 30 to 10. This is incompatible with convexity. A convex optimizer acting only on the target is therefore unnecessary and may be inappropriate; the monotone feasibility test is the useful structure.

Proposition 2 (Budget search)**.** *Assume that the full graph connects s and t, all edge costs are continuous, and all upper limits equal one. Extend the target cost to zero by setting $C(0) = 0$. For budget $B \geq 0$, the largest attainable target is*

$$\tau_B = \max\{\tau \in [0, 1] : C(\tau) \leq B\}. \tag{7}$$

Bisection produces a feasible lower target and an upper bound on τ_B separated by at most 2^{-r} after r steps.

Proof: Continuity makes the feasible target set closed. Monotonicity makes it an interval containing zero. At each bisection step replace the lower endpoint by the midpoint when its cost is at most B, and otherwise replace the upper endpoint. Feasibility of the lower endpoint is maintained and the interval length halves. □

Proposition 3 (Membership error)**.** *Assume that the full graph connects s and t, all upper limits are one, and the same edge cost functions are L Lipschitz on $[0,1]$ in both assessments. If two initial membership vectors in $[0,1]^E$ differ by at most ϵ in maximum norm, then, at every fixed target $\tau \in (0,1]$,*

$$|C_\mu(\tau) - C_{\tilde{\mu}}(\tau)| \leq (n-1)L\epsilon. \tag{8}$$

Proof: The positive part map is nonexpansive. Each transformed edge length changes by at most $L\epsilon$. A simple route contains at most $n-1$ edges. Comparing the optimal route under one vector with its cost under the other, in both directions, proves the result. □

This is a deterministic error bound. It does not require a probability distribution on membership assessments. Convex quadratic costs are a useful special case, but convexity plays no role in Theorem 1.

5| Algorithm

For a required target, perform the following steps.

(1) Remove edges whose upper membership limit is below the target.
(2) Evaluate (3) once for every remaining edge.
(3) Run Dijkstra's algorithm from s and retain predecessor and distance arrays.
(4) If t is unreachable, report infeasibility. Otherwise recover the terminal route and set only its required upgrade amounts.
(5) Check the route strength, total cost and distance inequalities.

Under the budget-search assumptions, for a target tolerance $\varepsilon \in (0,1)$, bisection repeats these steps at most $\lceil \log_2(1/\varepsilon) \rceil$ times. It is important to return the feasible lower target, not an arbitrary midpoint at termination. Fixed installation charges can also be included in a monotone edge cost, although discontinuous charges remove the continuity assumptions used in the stated budget existence argument.

6| Worked Network and Calculations

All data in this section are constructed for illustration. Vertices represent relay locations; membership measures link suitability under a common assessment scale. We use $c_e(x) = a_e x + b_e x^2$, upper limits one, and the edge data in Table 1.

Table 1. Complete input for the six vertex network. Costs are in normalized investment units.

Edge	μ_e	a_e	b_e
1–2	0.45	2	1
1–3	0.65	4	2
2–3	0.70	1	1
2–4	0.55	3	1
3–4	0.35	2	2
3–5	0.60	2	1
4–5	0.75	1	1
4–6	0.40	4	2
5–6	0.55	3	1

At target 0.70, the route $1,3,5,6$ requires upgrades 0.05, 0.10 and 0.15. Its cost is

$$\begin{aligned} C(0.70) &= (4(0.05) + 2(0.05)^2) + (2(0.10) + (0.10)^2) \\ &\quad + (3(0.15) + (0.15)^2) = 0.8875. \end{aligned} \tag{9}$$

The fewest hop comparator uses route $1,2,4,6$ after lexicographic tie breaking and costs 2.4150 at the same target. The widest current route happens to coincide with the optimized route over most of this example. Table 2 records that equality instead of suggesting a difference where none exists.

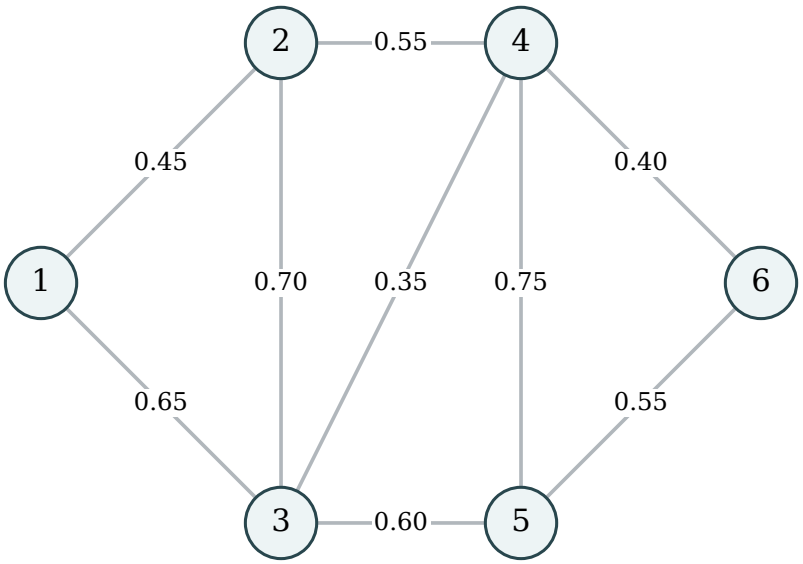

Figure 1. The illustrative network. Edge labels are initial memberships; terminals are vertices 1 and 6.

Table 2. Target costs and routes on the illustrative network.

Target	Optimum	Selected route	Fewest hops	Widest route
0.40	0.0000	1–2–4–6	0.0000	0.0000
0.50	0.0000	1–3–5–6	0.5225	0.0000
0.60	0.1525	1–3–5–6	1.3550	0.1525
0.70	0.8875	1–3–5–6	2.4150	0.8875
0.80	1.8975	1–3–5–6	3.5550	1.8975
0.90	2.9875	1–3–5–6	4.7750	2.9875

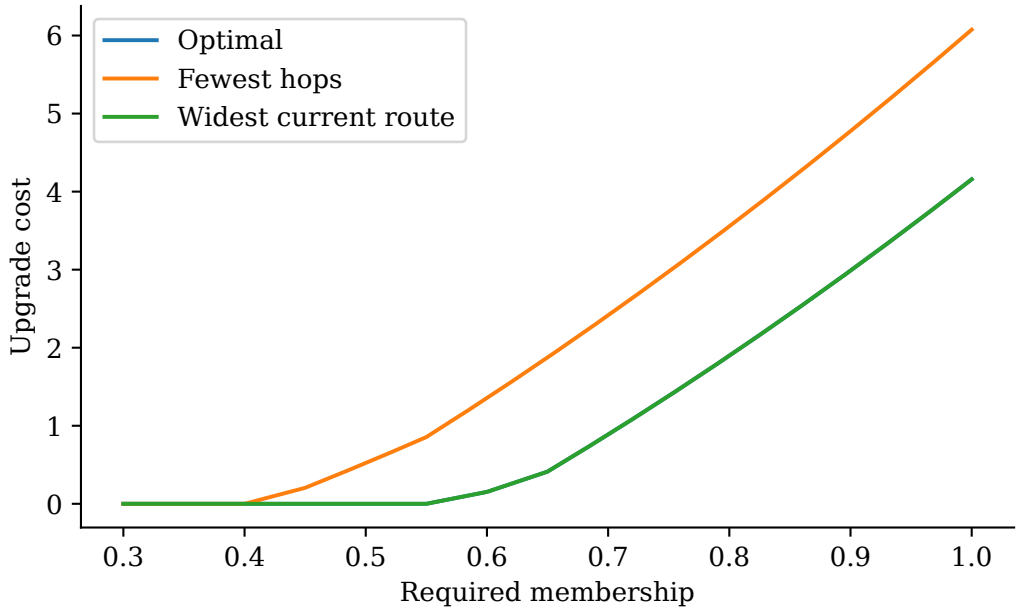

Figure 2. Cost as the required membership increases. Coincident curves indicate identical decisions in this example.

7| Computational Study

We generated 100 graphs with eight vertices. Edges $(i, i+1)$ form a fixed chain. Each other possible edge is independently included with probability 0.22. For each graph, memberships are independent uniform values in $[0.2, 0.8]$, linear cost coefficients in $[1, 5]$, and quadratic coefficients in $[0, 2]$. The generator is NumPy's `default_rng` with seeds $1100, \ldots, 1199$. Five targets yield 500 optimization instances.

The first comparator selects a route with the fewest edges and then upgrades that route. The second selects a route with the largest initial bottleneck; ties prefer fewer edges and then the lexicographically smaller vertex sequence. Both comparators use the same upgrade costs as the proposed method. All simple routes are enumerated independently to establish the exact optimum on these small graphs.

Table 3. Mean investment over 100 graphs at each target. The final columns count strict cost improvements, with tolerance 10^{-8}.

Target	Optimal mean	Hop mean	Widest mean	Beats hop (%)	Beats widest (%)
0.50	0.3228	0.5942	0.3920	55	21
0.60	0.7166	1.0999	0.8649	56	34
0.70	1.3202	1.7492	1.5672	50	38
0.80	2.0967	2.5077	2.4779	42	39
0.90	2.9714	3.3601	3.5469	35	45

The shortest path result and exhaustive minimum agree to the reported floating point precision in all 500 instances. At target 0.90, mean costs are 2.9714, 3.3601 and 3.5469 for the optimized, hop and widest rules. The optimized route is strictly less expensive than the widest rule in 45 of the 100 graphs. At lower targets, several routes already satisfy the condition at zero cost, explaining why strict improvement is not universal.

These comparisons measure the cost of a specified intervention. They do not demonstrate physical network performance, and they do not compare computational scaling on large graphs. The exact reduction, rather than the simulation, establishes optimality for every instance satisfying the model assumptions.

8| Discussion and Conclusion

The formulation is suitable when one adequate terminal route is the service requirement and edge improvements can be priced independently. It is less suitable when failures are probabilistic, routes must be disjoint, improvements have shared mobilization costs, or edge membership depends on other upgrades. In those settings the path optimality condition can remain a lower bound or a subproblem, but the exact identity requires further analysis.

The main result converts membership investment into additive route lengths. It separates the semantic choice of a fuzzy route strength from the combinatorial task of selecting an intervention. The sensitivity bounds explain how assessment errors affect cost, and the nonconvex example explains why target selection should use monotonicity rather than convexity. Future work may investigate shared investments for several terminal pairs and vertex dependent upper limits.

Data and Computation

The supplementary program reproduces every recorded instance. The base chain or cycle edges are listed first in vertex order, followed by additional edges in increasing endpoint order; the closing cycle edge is listed last among the base edges. Memberships and cost coefficients are drawn in separate arrays in that edge order. CSV files labelled `regenerated` contain the actual random seeds; the original files use a zero-based instance index in the first column. The worked inputs are fully listed. The simulation uses generated data only. The computation uses Python, NumPy and a heap implementation of Dijkstra's algorithm; the numerical outputs are supplied as CSV files. No external operational dataset, human participant data or measured field result is used.

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