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Facility Placement by Integrated Threshold Access in Fuzzy Networks

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Abstract

Facility placement on a fuzzy network is often based on one chosen membership threshold. A facility can then appear attractive or unattractive because of a small change in that threshold. We define an access objective that integrates coverage across all membership levels while limiting the number of edges in a service route. The integrated objective equals a weighted sum of the best bounded length bottleneck connection from each demand vertex to the selected facilities. This identity gives a finite computation using the maximum–minimum matrix product. The objective is monotone and submodular, so the classical greedy approximation result applies. We also derive deterministic bounds for errors in edge memberships and for transferring a selected facility set between two assessments. In 60 generated twelve vertex networks, greedy selection obtains a mean objective ratio of 0.9913 relative to exact enumeration, compared with 0.8840 for a weighted degree rule. The framework retains the meaning of threshold coverage while avoiding commitment to one threshold in the placement objective. The sensitivity result bounds the change in the objective by total demand times the maximum membership error. It also bounds the loss from transferring an optimal placement between two assessments. The model is intended for uncapacitated service planning with an explicit route length limit and a stated weighting of membership thresholds.

Keywords: Facility location, Fuzzy network, Threshold graph, Submodular function, Bounded length path, Maximum coverage.

1| Introduction

A service facility is useful to a demand vertex only when the connecting route meets an access requirement. In a network with graded edge suitability, this requirement is commonly represented by a threshold: retain edges whose membership is sufficiently large and ask which vertices can reach a facility. The answer depends on the threshold and on how long a service route is allowed to be.

One fixed threshold is suitable when a regulator or an operating rule specifies it. In exploratory planning, however, several levels may be relevant. Selecting facilities at one arbitrary level can discard information contained in the other edge memberships. We study the integral of threshold coverage as a placement objective, with a common bound on route length measured in edges.

The maximal covering location problem is well established [1]. Fuzzy graphs provide graded relations [2], [3], while facility location objectives of the form “sum of best service values” belong to classical submodular optimization [4]. The contribution here is an explicit threshold integral interpretation for a bounded length

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fuzzy connection, its matrix computation and a sensitivity analysis for membership assessment errors. The submodular approximation factor is inherited from established theory.

Classical location set covering seeks a smallest set of facilities that meets a specified time or distance standard at every demand point [5]. Maximal covering instead trades complete service for the largest covered demand under a facility limit [1]. Gradual covering models relax the binary service boundary by allowing coverage to decrease with distance [6]. Our score is also graded, but its service values are derived from bounded length bottleneck connectivity, rather than from a prescribed decay function of physical distance.

Fuzzy covering models distinguish several sources of imprecision. Davari, Fazel Zarandi and Hemmati [7] model fuzzy travel times using a fuzzy expected value objective; Davari, Fazel Zarandi and Turksen [8] study fuzzy coverage radii under the Hurwicz criterion. Arana-Jiménez, Blanco and Fernández [9] treat imprecision throughout the input data through fuzzy Pareto solutions and an equivalent mixed-binary multiobjective formulation. Here, demand weights and the facility limit are fixed, while edge memberships grade route access. Integrating over threshold levels produces a deterministic, real-valued coverage objective, not a fully fuzzy optimization problem or a probability of successful service.

2| Threshold Access Model

Let $G = (V, E, \mu)$ be an undirected fuzzy graph with vertex memberships one and edge memberships $\mu_e \in [0, 1]$. Demand weights $w_i \geq 0$ are given, with $W = \sum_i w_i > 0$. Every vertex is a candidate facility; a prescribed subset can be handled by restricting candidates. At most k facilities may be selected, where $1 \leq k \leq n = |V|$.

For threshold $t \in (0, 1]$, let G_t contain exactly the edges with $\mu_e \geq t$. Let $d_t(i, j)$ be unweighted graph distance in G_t, with infinity for disconnected pairs. For a nonnegative integer h, define

$$F(S) = \int_0^1 \sum_{i \in V} w_i \, \mathbf{1}\{\min_{s \in S} d_t(i, s) \leq h\} \, dt, \qquad F(\varnothing) = 0. \tag{1}$$

Thus $F(S)/W$ is the mean covered demand fraction over uniformly weighted threshold levels. The integral is a planning average over levels, not an expectation over random edge failures. An application can replace the uniform measure by a nonnegative policy weight on thresholds; the submodularity argument remains valid.

For each pair, set

$$q_{ij}^{(h)} = \max_{P: i \rightsquigarrow j, \ |P| \leq h} \min_{e \in P} \mu_e, \qquad q_{ii}^{(h)} = 1. \tag{2}$$

If no allowable route exists, the value is zero. We call $q_{ij}^{(h)}$ the bounded length connection strength. It measures the largest threshold at which access within h edges remains possible.

3| Exact Integral and Matrix Representations

Theorem 1 (Threshold identity)**.** *For every nonempty facility set,*

$$F(S) = \sum_i w_i \max_{s \in S} q_{is}^{(h)}. \tag{3}$$

Proof: For a fixed demand vertex, access to S within h edges at threshold t holds exactly when at least one allowable route has every edge membership at least t. This is equivalent to $t \leq \max_{s \in S} q_{is}^{(h)}$. The indicator is therefore one on an interval whose length is that maximum. Integrate this indicator and sum over the finite vertex set. □

Let A have diagonal entries one, $A_{ij} = \mu_{ij}$ on edges and zero otherwise. For matrices X, Y define

$$(X \otimes Y)_{ij} = \max_r \min\{X_{ir}, Y_{rj}\}. \tag{4}$$

This is the maximum–minimum product, rather than ordinary matrix multiplication. Initialize $Q^{(0)} = I$, with ones on the diagonal and zeros elsewhere, and compute $Q^{(r+1)} = Q^{(r)} \otimes A$.

Proposition 1 (Bounded length dynamic program)**.** $Q_{ij}^{(h)} = q^{(h)}{}_{ij}$. *Dense computation takes $O(hn^3)$ operations and $O(n^2)$ storage when successive matrices overwrite each other.*

Proof: At step zero, only a vertex is accessible to itself with strength one. Appending an edge takes the minimum of the existing route strength and the new edge membership, and choosing the best intermediate endpoint takes the maximum. Diagonal entries of one allow a route with fewer than h edges to be retained in subsequent products. Walks with repeated vertices can have cycles deleted without decreasing their minimum membership. Induction proves the identity. □

An alternative computation sorts the distinct positive edge memberships, forms each threshold graph, and computes reachability within h edges. Eq. (1) then becomes a finite weighted sum of coverage values. The two routes provide an independent implementation check. Threshold enumeration is especially useful when many edges share the same membership.

4| Submodularity and a Greedy Bound

Theorem 2 (Diminishing returns). *F is normalized, nonnegative, monotone and submodular. In particular, if $S \subseteq T$ and $v \notin T$, then*

$$F(S \cup \{v\}) - F(S) \geq F(T \cup \{v\}) - F(T). \tag{5}$$

Proof: Write $a_i(S) = \max_{s \in S} q_{is}^{(h)}$, taking $a_i(\varnothing) = 0$. The gain from adding v is

$$\Delta(v \mid S) = \sum_i w_i \max\{0, q_{iv}^{(h)} - a_i(S)\}. \tag{6}$$

All terms are nonnegative. Since $a_i(S) \leq a_i(T)$, each term of (6) for S is at least the corresponding term for T. Normalization and monotonicity follow from the same formula. □

Starting with the empty set, the greedy rule repeatedly selects the vertex of greatest marginal gain. Its performance is the standard cardinality constrained submodular bound [4]; the short proof specifies its application here.

Corollary 1 (Classical greedy factor). *If S_g contains k greedily selected vertices and S^* is optimal with at most k vertices, then*

$$F(S_g) \geq \left[1 - \left(1 - \frac{1}{k}\right)^k\right] F(S^*) \geq (1 - e^{-1}) F(S^*). \tag{7}$$

Proof: At iteration r, monotonicity and submodularity imply that the remaining gap $F(S^*) - F(S_r)$ is at most the sum of the marginal gains of the at most k elements of $S^* \setminus S_r$. The largest available gain is therefore at least one kth of that gap. After k steps the gap is at most $(1 - 1/k)^k F(S^*)$. □

This result does not imply that greedy is always optimal. It also depends on a cardinality constraint. Unequal facility costs with a budget constraint need an appropriate budgeted submodular algorithm, rather than a direct reuse of this proof. Budgeted maximum coverage [10] and monotone submodular maximization under a knapsack constraint [11] provide the relevant algorithmic extensions. Their approximation guarantees depend on the specified budget-aware algorithms and assumptions, not on applying the cardinality greedy rule unchanged.

5| Membership Sensitivity

Lemma 1 (Connection error). *For two assessments on the same graph, if corresponding edge memberships differ by at most ϵ, then $|q_{ij}^{(h)} - \tilde{q}_{ij}^{(h)}| \leq \epsilon$ for every pair.*

Proof: For each fixed route, the minimum of its memberships changes by at most ϵ. The maximum over the same finite family of allowable routes has the same bound. Zero length self access is unchanged. Edges absent from both graphs remain zero. □

Theorem 3 (Placement error and transfer). *For every facility set, $|F(S) - \tilde{F}(S)| \leq W\epsilon$. The two optimal objective values also differ by at most $W\epsilon$. If $\tilde{S}$ is an exact optimum under the perturbed assessment, then*

$$F(S^*) - F(\tilde{S}) \leq 2W\epsilon. \tag{8}$$

Proof: Taking a maximum over selected facilities preserves the bound of the lemma, and summing with nonnegative weights gives $W\epsilon$. Applying the bound to an optimum under each assessment proves the optimal value statement. Finally,

$$F(S^*) \leq \tilde{F}(S^*) + W\epsilon \leq \tilde{F}(\tilde{S}) + W\epsilon \leq F(\tilde{S}) + 2W\epsilon.$$

□

The bound explains a useful difference from a single threshold objective: the integrated objective changes continuously with membership values. Facility identities can still change abruptly when objective values are tied. Stable objective value and stable facility identity are distinct properties.

6| Algorithm and Worked Example

First construct $Q^{(h)}$. Maintain a vector a initialized to zero. At each of k steps, compute the gains in (6), select a largest gain vertex, and replace a_i by $\max\{a_i, q_{iv}^{(h)}\}$. With a precomputed dense Q, this costs $O(kn^2)$ operations. Fixing a vertex order makes ties reproducible. The final output consists of the selected facilities, the access strength at every demand vertex and their weighted sum.

The worked instance has twelve vertices, $h = 2$ and $k = 3$. It uses seed 3300 from the generator specified below. Figure 1 shows the network and Table 1 reports its demand contributions. The selected vertices are $3, 8, 4$, with the same set obtained by exact enumeration. The order reflects sequential greedy choices, not an ordering of facility importance.

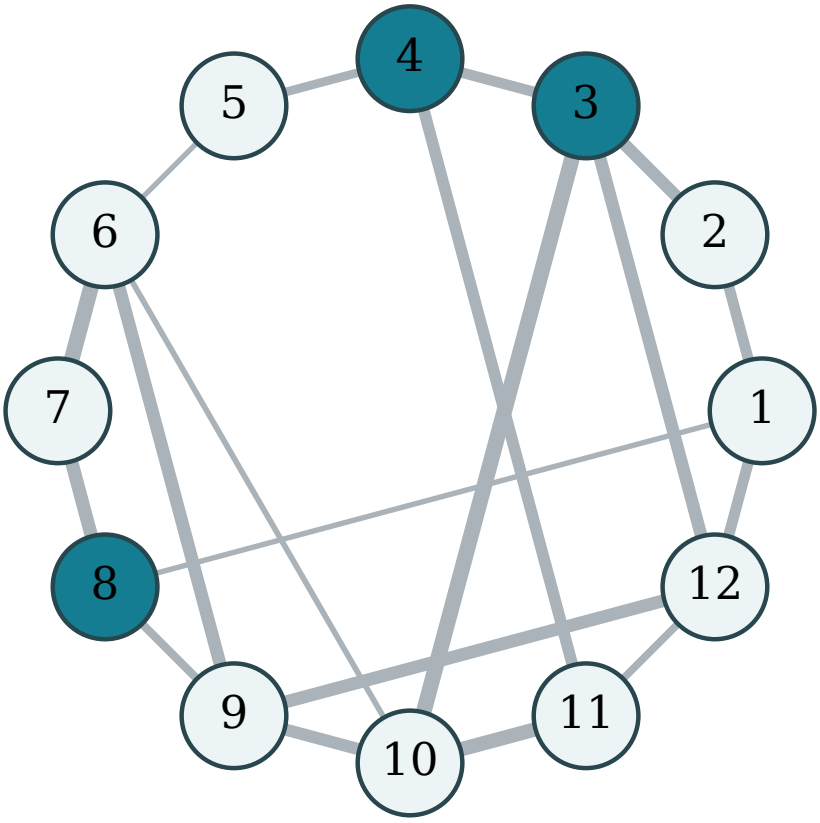

Figure 1. Constructed twelve vertex network. Highlighted vertices are the three greedy facilities; edge thickness represents membership.

Table 1. Demand and access contributions for the worked instance. Using unrounded strengths, the weighted sum is 36.1801 to four decimal places.

Vertex	Demand weight	Best strength to greedy facilities
1	5	0.5690
2	5	0.6625
3	6	1.0000
4	5	1.0000
5	2	0.4079
6	2	0.7459
7	1	0.7459
8	6	1.0000
9	4	0.6898
10	1	0.9085
11	6	0.8204
12	2	0.6898

Table 2. Facility choices on the worked instance.

Method	Selected vertices	Objective	Fraction of optimum
Greedy	3,8,4	36.1801	1.0000
Exact	3,4,8	36.1801	1.0000
Weighted degree	3,10,9	31.8086	0.8792
Threshold 0.60	10,6,1	34.7294	0.9599

To interpret one contribution, a demand weight of five and best connection strength 0.7 contribute 3.5 to F. This means that the demand is covered for threshold levels in an interval of length 0.7. It is not a statement that 70 percent of a randomly failing service is delivered.

7| Numerical Comparison

For each seed $3300, \ldots, 3359$, begin with the twelve cycle edges. Include each other possible undirected edge independently with probability 0.14. Draw memberships uniformly from $[0.15, 0.95]$, then demand weights uniformly from the integers 1 through 6. Use $h = 2$ and $k = 3$ in every instance. Enumerate all $\binom{12}{3} = 220$ sets to establish an exact optimum.

The weighted degree comparator selects the three largest sums of incident membership. It uses no demand weights in its ranking and is therefore a topology heuristic rather than an optimizer of F. The threshold comparator runs the same greedy coverage procedure on binary access at level 0.60, then evaluates its selection using the integrated objective. This isolates the effect of optimizing one threshold.

Table 3. Objective divided by the exact optimum over 60 generated graphs. SD is the sample standard deviation.

Method	Mean ratio	SD	Minimum	Exact optima
Greedy	0.9913	0.0148	0.9217	30
Weighted degree	0.8840	0.0620	0.7275	1
Threshold 0.60	0.9567	0.0357	0.8412	6

Greedy reaches the exact optimum in 30 instances. Its minimum observed ratio is 0.9217, which is higher than the theoretical worst case bound but does not improve that bound. The weighted degree and single threshold rules have lower mean ratios in this experiment. Results concern the specified generator and parameter values; they do not establish a universal empirical ranking for other graphs or service radii.

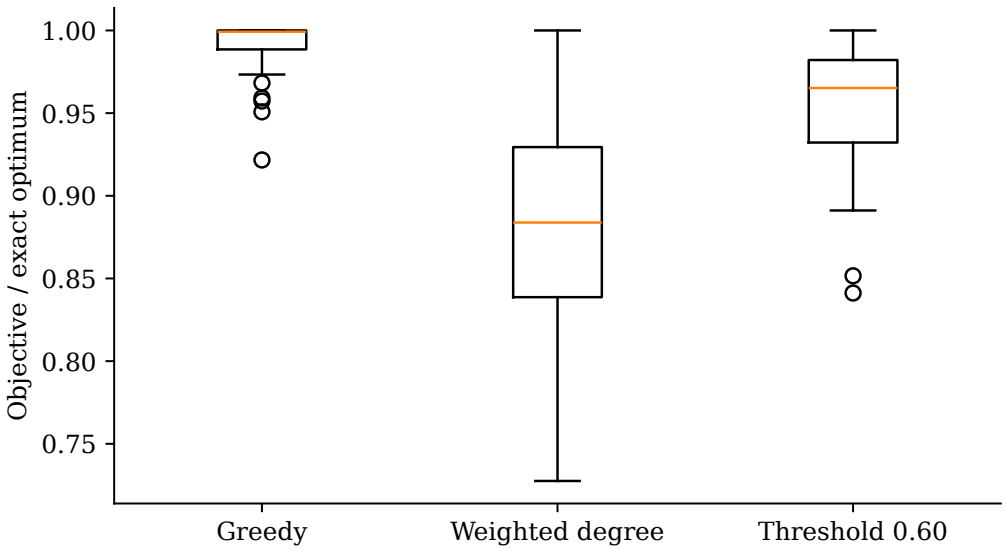

Figure 2. Distribution of objective ratios on the same 60 instances.

8| Discussion and Conclusion

Integrating threshold coverage gives a facility objective with three useful features: it preserves a direct access interpretation, can be evaluated by finite matrix operations, and has diminishing returns. Demand weighting changes which vertices matter while the bound on route length controls local service reach.

The model excludes congestion, facility capacities and path dependent travel time. The hop bound should be replaced by a suitable distance constraint when edges have unequal travel times. Uniform weighting of thresholds is a substantive planning choice; an application with an externally imposed threshold should retain that threshold rather than average it away. Approximate memberships also require a common scale before the deterministic perturbation bound has practical meaning.

The results connect fuzzy threshold graphs to an established facility optimization framework. Future work can study several service classes, facility capacities and policy selected threshold measures, with attention to whether submodularity survives those extensions.

Data and Computation

The supplementary program reproduces every recorded instance. The base chain or cycle edges are listed first in vertex order, followed by additional edges in increasing endpoint order; the closing cycle edge is listed last among the base edges. Memberships and cost coefficients are drawn in separate arrays in that edge order. CSV files labelled `regenerated` contain the actual random seeds; the original files use a zero-based instance index in the first column. All networks and demand weights are generated. The stated seeds and drawing order determine the complete instances. Exact enumeration is used only for validation on twelve vertices; the optimization method itself uses maximum–minimum dynamic programming and greedy marginal updates.

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